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KR-theory of compact Lie groups with group anti-involutions

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A B S T R A C T

Let G be a compact, connected, and simply-connected Lie group, equipped with an anti-involution a_G which is the composition of a Lie group involutive automorphism σ_G and the group inversion. We view (G, a_G) as a Real (G, σ_G)-space via the conjugation action. In this note, we exploit the notion of Real equivariant formality discussed in [9] to compute the ring structure of the equivariant KR-theory of G. In particular, we show that when G does not have Real representations of complex type, the equivariant KR-theory is the ring of Grothendieck differentials of the coefficient ring of equivariant KR-theory over the coefficient ring of ordinary KR-theory, thereby generalizing a result of Brylinski–Zhang's [8] for the complex K-theory case.

1. Introduction

Let G be a compact, connected and simply-connected Lie group, viewed as a G-space via the conjugation action. According to the main result of [8], the equivariant K-theory ring $K_G^*(G)$ is isomorphic to $\Omega_{R(G)/\mathbb{Z}}$, the ring of Grothendieck differentials of the complex representation ring of G over the integers (in fact, Brylinski–Zhang proved that this is true for $\pi_1(G)$ being torsion-free). Assuming further that G is equipped with an involutive automorphism σ_G, the author gave in [9] an explicit description of the ring structure of the equivariant KR-theory (cf. [2,3] and [6] for definition of KR-theory) $KR_G^*(G)$ by drawing on Brylinski–Zhang's result, Seymour's result on the module structure of $KR^*(G)$ (cf. [12]) and the notion of Real equivariant formality. $KR_G^*(G)$ in general has far more complicated ring structure and, among other things, is not a ring of Grothendieck differentials, as one would expect from Brylinski–Zhang's theorem. This is because in general the algebra generators of the equivariant KR-theory ring do not simply square to 0.

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In this note, we equip G instead with an anti-involution $a_G := \sigma_G \circ \mathrm{inv}$. Denoting the (G, σ_G)-space (G, a_G) by G^- for brevity, we compute the ring structure of $KR^*_G(G^-)$ following the idea of [9]. We find that there exists a derivation of the graded ring $KR^0_G(\mathrm{pt}) \oplus KR^{-4}_G(\mathrm{pt})$ taking values in $KR^1_G(G^-) \oplus KR^{-3}_G(G^-)$ (cf. Proposition 3.5) and that any element in the image of the derivation squares to 0 (see Propositions 3.3, 3.5 and 4.8(1), and compare with [9, Theorem 4.30, Proposition 4.31]). In particular,

Theorem 1.1. *If G does not have any Real representation of complex type with respect to σ_G, then the derivation in Proposition 3.5 induces the following ring isomorphism*

$$KR^*_G(G^-) \cong \Omega_{KR^*_G(\mathrm{pt})/KR^*(\mathrm{pt})}$$

Hence an anti-involution is the 'right' involution needed to generalize Brylinski–Zhang's result in the context of KR-theory. As a by-product, we also obtain the following

Corollary 1.2. *If G is a compact connected Real Lie group (not necessarily simply-connected) and X a compact Real G-space, then for any x in $KR^1_G(X)$ or $KR^{-3}_G(X)$, $x^2 = 0$.*

Note that graded commutativity only implies that x^2 is 2-torsion.

Throughout this note, G is a compact, connected and simply-connected Lie group unless otherwise specified. We sometimes omit the notation for the involution when it is clear from the context that a Real structure is implicitly assumed.

2. Background

In this section, we recall some relevant definitions and results from [8] and [9] needed in this note. We refer the reader to [2,3] and [6] for the basic definition of (equivariant) KR-theory, which we shall omit here.

Definition 2.1. Let G be a compact Lie group equipped with an involutive automorphism σ_G, i.e. a Real compact Lie group, and X a finite CW-complex equipped with an involution.

(1) (cf. [9, Proposition 2.29]) Let $c : KR^*_G(X) \to K^*_G(X)$ be the *complexification map* which forgets the Real structure of Real vector bundles, and $r : K^*_G(X) \to KR^*_G(X)$ be the *realification map* defined by

$$[E] \mapsto [E \oplus \sigma^*_X \sigma^*_G \overline{E}]$$

where σ^*_G means twisting the original G-action on E by σ_G.

(2) (cf. [9, Definitions 2.1 and 2.5]) Let $\delta : R(G) \to K^{-1}(G)$ be the derivation of $R(G)$ taking values in the $R(G)$-module $K^{-1}(G)$ (the module structure is realized by the augmentation homomorphism), where $\delta(\rho)$ is represented by the complex of vector bundles

$$\begin{aligned} &0 \to G \times \mathbb{R} \times V \to G \times \mathbb{R} \times V \to 0, \\ &(g, t, v) \mapsto (g, t, -t\rho(g)v) \text{ if } t \geq 0, \\ &(g, t, v) \mapsto (g, t, tv) \text{ if } t \leq 0. \end{aligned}$$

We define $\delta_G : R(G) \to K^{-1}_G(G)$ similarly. $\delta_G(\rho)$ is represented by the same complex of vector bundles where G acts on $G \times \mathbb{R} \times V$ by $g_0 \cdot (g_1, t, v) = (g_0 g_1 g_0^{-1}, t, \rho_V(g_0)v)$.

(3) Let σ_n be the class of the standard representation of $U(n)$ in $R(U(n))$. Let T be the standard maximal torus of $U(n)$. Let $\sigma_{\mathbb{R}}$ be the complex conjugation on $U(n)$, T, $U(n)/T$ or $U(\infty)$. Let $\sigma_{\mathbb{H}}$ be the symplectic