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An uncountable homology group, where each element is an infinite product of commutators

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ABSTRACT

In 1986 Umed Karimov constructed a space as a one-point compactification of a CW-complex, where each homotopy class of loops can be represented as an infinite product of commutators. We prove Karimov's conjecture that the first singular homology group of this space is uncountable. The proof uses methods of combinatorial group theory for one-relator groups and methods of geometric topology for smoothing singularities in van-Kampen diagrams corresponding to infinite words.

1. Introduction

In [16] Umed Karimov constructed a space which, up to homotopy, can be described as a one-point compactification of a 2-dimensional CW-complex where every basic loop of the one-skeleton can be homotoped into a commutator of other loops. Therefore, in this case, every path in the space is homotopic to an infinite product of commutators (cf. Remark 2.6). In 2002, in private communications with the second author, Karimov expressed his interest in understanding the homology group of this space and unveiled that he could show that this group is nontrivial. Furthermore, he conjectured that this group is big and that he would like to know a proof of that. In this paper we prove that the first homology group of Karimov's space is indeed uncountable (cf. Theorems 2.5 and 7.2).

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Determining and describing the algebraic invariants of spaces which are non-triangulable, but have instead an accumulation of small parts that generate nontrivial elements in their homology- or homotopy groups (i.e. of "wild" spaces) is in general a non-schematic and nontrivial task. Many tools of algebraic topology (such as van-Kampen's Theorem, Mayer–Vietoris' Theorem and the excision theorem) cannot be applied in these cases, since they depend on local conditions which are not satisfied for such spaces. Also, the axioms of homology theory can only be expected to determine the homology groups of triangulable spaces (cf. [24, 4.8.10]).

The Hawaiian Earrings (cf. Fig. 1) form the simplest non-triangulable space that already results in an unexpected behavior of its algebraic invariants. Therefore, it is very natural to use the Hawaiian Earrings as building blocks for construction of spaces with more interesting properties. Apparently the first application of this type was done by Griffiths in [10, Section 3] and [11, Introduction], where he constructed a space with nontrivial fundamental group that is a one-point union of spaces with trivial fundamental groups, cf. [22, End of Introduction], [12].

Karimov's space is also constructed with the help of Hawaiian Earrings, but the relations used for that are more complicated as in the case of Griffiths' space. The one-skeleton of Karimov's space is homotopy equivalent to the Hawaiian Earrings $\mathbb{HE}$, and the fundamental and the first homology groups of Karimov's space can therefore be described as factor groups of the fundamental group of the Hawaiian Earrings. An interesting property of Karimov's space (as we show) is that the first singular homology group of this space is uncountable and that each element of this group can be naturally expressed as an infinite product of commutators.

Already the fundamental group of the Hawaiian Earrings provides the psychological surprise that it is not a free group (first proven in [15, p. 80], see also [23], [4, Corollary 4.17], [26, Corollary 4.5]) and that it is neither the direct nor the inverse limit of finitely generated free groups, but it is a group which sits somehow in between. Several authors (cf. Remark 2.2) investigated this fundamental group, and now it is absolutely clear which infinite products of basic loops in $\mathbb{HE}$ are admissible, i.e. can be realized by continuous paths.

The algebraic structure of the homology group of the Hawaiian Earrings has been determined in [7, Theorem 3.1], however this description does not relate the elements in the homology group to concrete geometric realizations of cycles. An alternative approach to understand the homology group of the Hawaiian Earrings is contained in [4, Section 4] and [3]. In particular, [3] describes transformations of infinite words such that two admissible infinite products w_1 and w_2 determine the same element in the homology group if and only if w_1 can be carried to w_2 by application of a finite number of such transformations.

However, up to this moment, there is only few papers on the homology- or fundamental groups of two-dimensional wild spaces with nontrivial relations. In [2] Bogley and Sieradski tried to develop a theory of "Omega-presentations" which only works for metric one-vertex complexes and is based on their concept of Omega-groups, (i.e. on a generalization of groups that also allows infinite products). In [9] Paul Fabel tried to describe the fundamental group of the Harmonic Archipelago, and in [3] we have investigated the fundamental and homology groups of Griffiths' space. The authors in [3,9] and [4] use the fact that the relations in these groups are fairly simple; these relations can be described by identifications of symbols for the Harmonic Archipelago, and by deletion of symbols in case of Griffiths' space, and by permutations of symbols if it comes to describe the homology groups.

To our best knowledge, this paper is the first which studies the homology group of a wild space with more complicated relations for generators of its fundamental group than just deletion, permutation or identification of generators. We show that the homology group of Karimov's space is uncountable, by using a combination of methods of geometric topology and of combinatorial group theory. In particular, we use a result on normal closures of elements in free groups which is equivalent to Magnus' Freiheitssatz [21] on one-relator groups.

Of course, our main result implies that the fundamental group of Karimov's space is uncountable. This fact also follows from [8] by combining its Theorem 1.1 and Lemma 3.1. Due to this lemma, our main result