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On local compactness of the subspace $FP_2(X)$ of $FP(X)$ [☆]

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ABSTRACT

In this paper, it is shown that if X is homeomorphic to the topological sum of a Tychonoff space Y and a discrete space D, then $FP_2(X)$ is locally compact if and only if $FP_2(Y)$ is locally compact. Especially, the space $FP_2(X)$ is locally compact at its every point distinct from the identity e if X is homeomorphic to the topological sum of a compact space and a discrete space. Finally, a few questions about free paratopological groups are posed.

1. Introduction

A topological group is a group G with a topology such that the multiplication mapping of $G \times G$ to G is jointly continuous and the inverse mapping of G on itself is also continuous. A paratopological group is a group G with a topology such that the multiplication mapping of $G \times G$ to G is jointly continuous. It is well known that paratopological groups are good generalization of topological groups. The absence of continuity of inversion, the typical situation in paratopological groups, makes the study in this area very different from that in topological groups.

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In 1941, free topological groups in the sense of A. Markov were introduced [12]. Until now, there have been many topologists concentrating on this topic, see e.g. [1,8,10] and so on. Especially, in 2003, P. Nickolas and M. Tkachenko studied local compactness in free topological groups [13]. As in free topological groups, in 2002, S. Romaguera, M. Sanchis, and M. Tkachenko [16] introduced free paratopological groups on arbitrary topological spaces and discussed some of their topological properties. As they wrote in [16], free paratopological groups are probably also a tool for solving some problems in the theory of paratopological groups. Under the motivation of some open problems about free paratopological groups [1,16], publications in the field of free paratopological groups have increasingly emerged [2–4,9,11,14,15]. The study about free paratopological groups has been an active research subject.

Not long ago, in a private communication, F. Lin informed the author he had showed that if X is a μ-space and the subspace $FP_2(X)$ of free paratopological group $FP(X)$ is locally compact, then X is homeomorphic to the topological sum of a compact space and a discrete space. Also, the following Question 1.1 was posed by F. Lin.

Question 1.1. *Is the subspace $FP_2(X)$ of free paratopological group $FP(X)$ locally compact, provided that X is homeomorphic to the topological sum of a compact space and a discrete space?*

Around Question 1.1, we shall show that if X is homeomorphic to the topological sum of a Tychonoff space Y and a discrete space D, then $FP_2(X)$ is locally compact if and only if $FP_2(Y)$ is locally compact. In particular, the space $FP_2(X)$ is locally compact at its every point distinct from the identity e if X is homeomorphic to the topological sum of a compact space and a discrete space.

In this paper, $F_a(X)$ algebraically denotes the free group on non-empty set X and e is the identity of $F_a(X)$. Every $g \in F_a(X)$ distinct from e has the form $g = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where $x_1, \cdots, x_n \in X$ and $\varepsilon_1, \cdots, \varepsilon_n = \pm 1$. This expression or word for g is called reduced if it contains no pair of consecutive symbols of the form xx^{-1} or $x^{-1}x$ and we say in this case that the length $l(g)$ of g equals to n. For every non-negative integer n, denote by $FP_n(X)$ the subspace of the free paratopological group $FP(X)$ that consists of all words of reduced length $\leq n$ with respect to the free basis X.

In what follows, the subspace X of free paratopological group $FP(X)$ is assumed to be Hausdorff, unless stated otherwise. For some terminology unstated here, readers may refer to [1,5].

2. Preliminaries

Definition 2.1. ([16]) Let X be a subspace of a paratopological group G. Suppose that

(1) the set X generates G algebraically, that is, $\langle X \rangle = G$; and
(2) every continuous mapping $f : X \to H$ of X to an arbitrary paratopological group H extends to a continuous homomorphism $\hat{f} : G \to H$.

Then G is called the Markov free paratopological group (briefly, free paratopological group) on X and is denoted by $FP(X)$.

Remark 2.2. It has been shown that the topology of $FP(X)$ is the finest paratopological group topology on the abstract free group $F_a(X)$ of X which induces the original topology on X [16]. Both X and X^{-1} are closed in $FP(X)$ and X^{-1} is a discrete subspace [3].

Definition 2.3. ([6,7]) A quasi-uniformity $\mathcal{U}$ on a set X is a filter on $X \times X$ such that

(1) each member U of $\mathcal{U}$ contains the diagonal $\triangle_X = \{(x, x) : x \in X\}$ of X;
(2) for each $U \in \mathcal{U}$, there exists a $V \in \mathcal{U}$ such that $V \circ V = \{(x, z) \in X \times X : \text{there is } y \in X \text{ such that } (x, y) \in V \text{ and } (y, z) \in V\} \subset U$.