Topology and its Applications 198 (2016) 22–33

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On topological shape homotopy groups

Tayyebe Nasri, Fateme Ghanei, Behrooz Mashayekhy *, Hanieh Mirebrahimi

Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran

ARTICLE INFO

Article history:
Received 10 May 2015
Received in revised form 9 November 2015
Accepted 9 November 2015
Available online 17 November 2015

MSC:
55Q07
55P55
54C56
54H11
18A30

Keywords:
Topological shape homotopy group
Shape group
Topological group
Inverse limit

ABSTRACT

In this paper, using the topology on the set of shape morphisms between arbitrary topological spaces X, Y, $Sh(X,Y)$, defined by Cuchillo-Ibanez et al. in 1999, we consider a topology on the shape homotopy groups of arbitrary topological spaces, which make them Hausdorff topological groups. We then exhibit an example in which $\check{\pi}_k^{top}$ succeeds in distinguishing the shape type of X and Y while $\check{\pi}_k$ fails, for all $k \in \mathbb{N}$. Moreover, we present some basic properties of topological shape homotopy groups, among them commutativity of $\check{\pi}_k^{top}$ with finite product of compact Hausdorff spaces. Finally, we consider a quotient topology on the kth shape group, induced by the kth shape loop space and show that it coincides with the above topology.

1. Introduction and motivation

Morón et al. [20] gave a complete, non-Archimedean metric (or ultrametric) on the set of shape morphisms between two unpointed compacta (compact metric spaces) X, Y, written $Sh(X,Y)$. Then they mentioned that this construction can be translated to the pointed case. Consequently, as a particular case, they obtained a complete ultrametric that induces a norm on the shape groups of a compactum Y and then presented some results on these topological groups [21]. Also, Cuchillo-Ibanez et al. [7] constructed many generalized ultrametrics in the set of shape morphisms between topological spaces and obtained semivaluations and valuations on the groups of shape equivalences and kth shape groups. On the other hand, Cuchillo-Ibanez et al. [8] introduced a topology on the set $Sh(X,Y)$, where X and Y are arbitrary topological spaces,

* Corresponding author.
E-mail addresses: tnasri72@yahoo.com (T. Nasri), fatemeh.ghanei91@gmail.com (F. Ghanei), bmashf@um.ac.ir (B. Mashayekhy), h_mirebrahimi@um.ac.ir (H. Mirebrahimi).

http://dx.doi.org/10.1016/j.topol.2015.11.003
0166-8641/

in such a way that it extended topologically the construction given in [20]. Also, Moszyńska [22] showed that the kth shape group $\check{\pi}_k(X,x)$, $k \in \mathbb{N}$, is isomorphic to the set $Sh((S^k,*),(X,x))$ consists of all shape morphisms $(S^k,*) \to (X,x)$ with a group operation, for all compact Hausdorff space (X,x). Note that, Bilan [2] mentioned that this fact is true for all topological spaces. In this paper, we consider the latter topology on the set of shape morphisms between pointed spaces, consequently we obtain a topology on the shape groups of arbitrary spaces. We show that the kth shape group $\check{\pi}_k(X,x)$, $k \in \mathbb{N}$, with the above topology is a Hausdorff topological group, denoted by $\check{\pi}_k^{top}(X,x)$. We then exhibit an example in which $\check{\pi}_k^{top}$ succeeds in distinguishing the shape type of X and Y while $\check{\pi}_k$ fails, for all $k \in \mathbb{N}$. In fact, we exhibit two topological spaces X and Y, whose kth shape groups are isomorphic while their topological kth shape groups are not isomorphic, for all $k \in \mathbb{N}$. Also, we show that $\check{\pi}_k^{top}$ preserves the product $X \times Y$ provided X and Y are compact Hausdorff spaces. Moreover, we obtain some topological properties of these groups and we study some properties in shape as movability. We show that movability can be transferred from a pointed topological spaces (X,x) to $\check{\pi}_k^{top}(X,x)$, under some conditions. Also, we show that if $X \in HPol$, then $\check{\pi}_k^{top}(X,x)$ is discrete. Moreover, if $\mathbf{p} : X \longrightarrow \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is an HPol-expansion of X and $\check{\pi}_k^{top}(X,x)$ is discrete, then $\check{\pi}_k^{top}(X,x) \leq \check{\pi}_k^{top}(X_\lambda, x_\lambda)$, for some $\lambda \in \Lambda$ and for all $k \in \mathbb{N}$.

Endowed with the quotient topology induced by the natural surjective map $q : \Omega^k(X,x) \to \pi_k(X,x)$, where $\Omega^k(X,x)$ is the kth loop space of (X,x) with the compact-open topology, the familiar homotopy group $\pi_k(X,x)$ becomes a quasitopological group which is called the quasitopological kth homotopy group of the pointed space (X,x), denoted by $\pi_k^{qtop}(X,x)$ (see [3–5,14]).

Biss [3] proved that $\pi_1^{qtop}(X,x)$ is a topological group. However, Calcut and McCarthy [6] and Fabel [10] showed that there is a gap in the proof of [3, Proposition 3.1]. The misstep in the proof is repeated by Ghane et al. [14] to prove that $\pi_k^{qtop}(X,x)$ is a topological group [14, Theorem 2.1] (see also [6]).

Calcut and McCarthy [6] showed that $\pi_1^{qtop}(X,x)$ is a homogeneous space and more precisely, Brazas [4] mentioned that $\pi_k^{qtop}(X,x)$ is a quasitopological group in the sense of [1].

Calcut and McCarthy [6] proved that for a path connected and locally path connected space X, $\pi_1^{qtop}(X)$ is a discrete topological group if and only if X is semilocally 1-connected (see also [4]). Pakdaman et al. [25] showed that for a locally $(k-1)$-connected space X, $\pi_k^{qtop}(X,x)$ is discrete if and only if X is semilocally n-connected at x (see also [14]). Fabel [10,11] and Brazas [4] presented some spaces for which their quasitopological homotopy groups are not topological groups. Moreover, despite of Fabel's result [10] that says the quasitopological fundamental group of the Hawaiian earring is not a topological group, Ghane et al. [15] proved that the topological kth homotopy group of a k-Hawaiian like space is a prodiscrete metrizable topological group, for all $k \geq 2$.

For an HPol-expansion $\mathbf{p} : X \to (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of X, Brazas [4] introduced a topology on $\check{\pi}_k(X)$. Here we denote it by $\check{\pi}_k^{inv}(X)$. He mentioned that since $\pi_k^{qtop}(X_\lambda)$ is discrete, for all $k \in \mathbb{N}$, one can define the kth topological shape group of X as the limit $\check{\pi}_k^{inv}(X) = \lim\limits_{\leftarrow} \pi_k^{qtop}(X_\lambda)$ which is an inverse limit of discrete groups and so it is a Hausdorff topological group [4, Remark 2.19]. In this paper, we prove that $\check{\pi}_k^{inv}(X,x) \cong \check{\pi}_k^{top}(X,x)$ as topological groups, for every pointed topological space (X,x). The second view at this topology implies some results which, it seems, are not obtained by the first view.

Finally, for an HPol$_*$-expansion $\mathbf{p} : (X,x) \to ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ of a pointed topological space (X,x), we define the kth shape loop space as a subspace of $\prod_{\lambda\in\Lambda} \Omega^k(X_\lambda, x_\lambda)$ and consider a quotient topology on the kth shape group induced by the kth shape loop space. Then we show that this quotient topology on the kth shape group coincides with the topology of $\check{\pi}_k^{top}$.

2. Preliminaries

In this section, we recall some of the main notions concerning the shape category and pro-HTop (see [18]). Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be two inverse systems in HTop. A *pro-morphism* of inverse