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Factorization of bijections onto ordered spaces

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ABSTRACT

We identify a class of subspaces of ordered spaces $\mathcal{L}$ for which the following statement holds: If $f : X \to L \in \mathcal{L}$ is a continuous bijection of a zero-dimensional space X, then f can be re-routed via a zero-dimensional subspace of an ordered space that has weight not exceeding that of L.

1. Introduction

The paper studies factorization of continuous bijections onto subspaces of ordered spaces. In [4], Mardesic proved, in particular, that any continuous map of an n-dimensional compactum into a metric compact space can be re-routed via a metric compactum of dimension at most n. Deep research has been inspired by this result. We would like to mention a theorem of Pasynkov [5] that any continuous map of a Tychonoff Ind-n space into a metric space M admits a factorization with the middle space of Ind-dimension at most n and weight at most $w(M)$. This result implies, in particular, that if a space of Ind-dimension n admits a continuous injection into a metric space, then it admits a continuous injection into a metric space of Ind-dimension at most n. The paper is devoted to the following general problem:

Problem. Let a space X of dimension n admit a continuous bijection onto a space with property P. Does X admit a continuous bijection onto a space of dimension at most n and with property P?

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In this paper we are interested in inductive "ind" dimension. We first observe that for any $n > 2$ there exists a zero-dimensional space X that admits a continuous injection into $\mathbb{R}^n$ but not into $\mathbb{R}^{n-1}$. We then observe that, if a zero-dimensional space continuously injects into $\mathbb{R}$, then it admits a continuous injections into the Cantor Set. The latter observation led the authors to the main result of this paper. To state the result, let $\mathcal{L}$ be the class of all subspaces of ordered spaces that have a σ-disjoint π-base. Note that a subspace L of an ordered space is in $\mathcal{L}$ if and only if L has a dense subset $\cup_n X_n$, where each X_n is a discrete (in itself) subspace. In our main result, Theorem 3.3, we prove that *if $f : X \to L \in \mathcal{L}$ is a continuous bijection of a zero-dimensional space X, then f can be re-routed via a zero-dimensional subspace M of an ordered space of weight at most that of L.* One may wonder if the class $\mathcal{L}$ can be replaced by the class of all ordered spaces and their subspaces. The authors do not have an example to justify the restriction on the class $\mathcal{L}$ but believe that there is a good chance for an example.

In notation and terminology, we will follow [1]. Spaces of ind-dimension 0 are called *zero-dimensional*. A linearly ordered space (abbreviated as LOTS) is one with the topology induced by some order. A subspace of a LOTS is a generalized ordered space (abbreviated as GO). It is a known and very useful fact that a Hausdorff space L is a GO-space, if its topology can be generated by a collection of convex sets with respect to some order on L (see, for example, [2]). A subset A of an ordered set $\langle S, \prec\rangle$ is *$\prec$-convex* if it is convex with respect to $\prec$. A collection $\mathcal{U}$ of non-empty open sets of a space X is *a π-base* of X if for any non-empty open set O in X there exists $U \in \mathcal{U}$ such that $U \subset O$. All spaces are assumed Tychonoff. For the purpose of readability we will occasionally resort to an informal argument.

2. Motivation

Ample research has been done to describe spaces that admit continuous injections into metric spaces. Assume that a space X has a certain dimension and admits a continuous injection into a metric space. It is natural to wonder if X admits a continuous injection into a metric space of the same dimension. As mentioned in the introduction, if dimension is in the sense of Ind or dim, then the affirmative answer is a simple corollary of the mentioned Pasynkov factorization theorem. In case of ind-dimension, additional analysis may be required. We start with examples.

Example 2.1. For each $N > 1$ there exists a separable space X of $ind(X) = 0$ that admits a continuous injection into $\mathbb{R}^N$ but not into $\mathbb{R}^{N-1}$.

Construction Put $D = \{\langle q_1, \dots, q_N\rangle : q_i \in \mathbb{Q}\}$ and $P = \mathbb{R}^N \setminus D$. The underlying set for our space is $\mathbb{R}^N = P \cup D$. We will define a new topology $\mathcal{T}$ on $\mathbb{R}^N$ so that $X = \langle \mathbb{R}^N, \mathcal{T}\rangle$ has desired properties. When $\mathbb{R}^N$ is used with the Euclidean topology, we will refer to it by $\mathbb{R}^N$. In our new topology, points of D are declared isolated. Base neighborhoods at points of P will be defined in two stages.

Stage 1. For each map $f : D \to \mathbb{R}^{N-1}$, put

$$P_f = \{x \in P : f \text{ cannot be continuously extended to } D \cup \{x\}\},$$

where continuity at $x \in P$ is considered with respect to the Euclidean topology and D is regarded as discrete. Next let F be the set of all functions f from D to $\mathbb{R}^{N-1}$ such that $|P_f| = 2^\omega$. For each $f \in F$, fix $x_f \in P_f$ so that $x_f \neq x_g$ for distinct $f, g \in F$. This can be done since $|F| = |P_f| = 2^\omega$ for each $f \in F$. Let us define a base neighborhood at x_f, for a fixed $f \in F$. Recall that f cannot be continuously extended to x_f with respect to the Euclidean topology at x_f. Therefore, there exists a sequence $\langle x_{f,n}\rangle_n$ of elements of D that converges to x_f in $\mathbb{R}^N$ such that $\langle f(x_{f,n})\rangle_n$ is not converging in $\mathbb{R}^{N-1}$. Sets in form $\{x_f\} \cup \{x_{f,i} : i > n\}$ will be base neighborhoods at x_f in our new topology. This completes Stage 1.