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# Equicontinuity on semi-locally connected spaces ☆

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We prove that a homeomorphism of a semi-locally connected compact metric space is equicontinuous if and only if it satisfies the following property (CW): the distance between the iterates of a given point and a given subcontinuum not containing that point is bounded away from zero. This equivalence is false for general compact metric spaces. We also prove that a homeomorphism with the property (CW) satisfies that the set of automorphic points contains those points where the space is not semi-locally connected.



## 1. Introduction

A homeomorphism $f : X \to X$ of a compact metric space $X$ is *distal* if

$$\inf_{n\in\mathbb{Z}} d(f^n(x), f^n(y)) > 0$$

for all distinct points $x, y \in X$. The distal homeomorphisms were introduced by Hilbert as a generalization of rigid group of motions [3,6].

We can generalize this definition by replacing $y$ above by a compact subset $K$ (not containing $x$ of course); and $d(f^n(x), f^n(y))$ by $dist(f^n(x), f^n(K))$ where

$$dist(p, B) = \inf\{d(p, b) : b) \in B\}, \qquad \forall p \in X, B \subset X.$$

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However, the resulting generalization turns out to be equivalent to equicontinuity (by Lemma 2.5 below). Recall that $f$ is *equicontinuous* if for every $\epsilon > 0$ there is $\delta > 0$ such that $d(f^n(x), f^n(y)) \leq \epsilon$ whenever $x, y \in X$ satisfy $d(x, y) \leq \delta$.

But the question arises: What if we replace $y$ by another type of compact subsets like, for instance, a subcontinuum $C$ not containing $x$? Recall that a *subcontinuum* is a nonempty compact connected subset of $X$. The continuum theory [12] has been considered elsewhere in the study of dynamical systems [1,10]. The question suggests the study of homeomorphisms $f : X \to X$ with the following property:

(CW) $\inf_{n\in\mathbb{Z}} dist(f^n(x), f^n(C)) > 0$ for every $x \in X$ and every subcontinuum $C$ of $X$ with $x \notin C$.

In our first result we will give topological conditions characterizing equicontinuity in terms of this property. For this we shall use the following definition by Whyburn [14] (see also Jones [9]).

**Definition 1.1.** We say that a metric space $X$ is *semi-locally connected at* $p \in X$ if for every open neighborhood $U$ of $p$ there is an open neighborhood $V \subset U$ of $p$ such that $X \setminus U$ is contained in the union of finitely many connected components of $X \setminus V$. We denote by $X^{slc}$ the set of points at which $X$ is semi-locally connected. We say that $X$ is *semi-locally connected* if $X^{slc} = X$.

The class of semi-locally connected spaces is broad enough to include properly the locally connected ones (like manifolds or more structured spaces). In these spaces we shall obtain the following equivalence.

**Theorem 1.2.** *A homeomorphism of a semi-locally connected compact metric space is equicontinuous if and only if it satisfies (CW).*

In particular, since there are distal homeomorphisms on the two-disk which are not equicontinuous (e.g. Chapter 5 in [3]), and the two-disk is locally connected, we obtain the following corollary.

**Corollary 1.3.** *There are distal homeomorphisms which do not satisfy (CW).*

Notice that (CW) is also equivalent to equicontinuity on certain non-semi-locally connected spaces like the totally disconnected ones (see Corollary 1.9 in [4]). But in general (CW) is not equivalent to equicontinuity as in the following counterexample.

**Example 1.4.** Define $X = \{re^{i\theta} \in \mathbb{R}^2 : 0 \leq \theta \leq 2\pi, r \in C\}$ where $C$ is the ternary Cantor set in $[1, 2]$. Then, the map $f : X \to X$ defined by $f(re^{i\theta}) = re^{i(\theta+2\pi r)}$ is a homeomorphism of $X$. This homeomorphism is an irrational rotation on the circle with radius $r$ for irrational $r$, while it is a periodic rotation for rational $r$. Using this we can see that $f$ is not equicontinuous. Since every subcontinuum of $X$ is contained in one of the circles $\{re^{i\theta} : 0 \leq \theta \leq 2\pi\}$ $(r \in C)$ in which the action is an isometry, we obtain that $f$ satisfies (CW).

This counterexample motivates the search of similarities between (CW) and the equicontinuous property. For instance, since every equicontinuous homeomorphism is uniformly rigid [7], it is natural to ask if every homeomorphism satisfying (CW) is uniformly rigid (or at least rigid). Another question comes from the following definition by Veech [13] (see also [5]):

**Definition 1.5.** If $f : X \to X$ is a homeomorphism of a compact metric space $X$, a point $x \in X$ is called *almost automorphic* if for every sequence $n_i \in \mathbb{Z}$ with $f^{n_i}(x) \to y$ for some $y \in X$ it holds that $f^{-n_i}(y) \to x$. We denote by $A(f)$ the set of almost automorphic points of $f$.

Theorem 3 in [5] implies that every equicontinuous homeomorphism $f : X \to X$ of a compact metric space $X$ satisfies $A(f) = X$. Is this property true (or at least $A(f) \neq \emptyset$) for every homeomorphisms $f$