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## Nonseparable C(K)-spaces can be twisted when K is a finite height compact $\overset{\bigstar, \bigstar \bigstar}{\leftarrow}$

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ABSTRACT

nonmetrizable compact set?

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## 1. Introduction

A general problem in Banach space theory is to determine, given two Banach spaces Y and Z, the existence and properties of Banach spaces X containing Y such that X/Y = Z. The space X is called a twisted sum of Y and Z. When only the trivial situation  $X = Y \oplus Z$  is possible one obtains an interesting structure result asserting that every copy of Y in a space X such that X/Y = Z must be complemented. When some  $X \neq Y \oplus Z$  as above exists, one obtains a usually exotic and interesting space with unexpected properties; perhaps the perfect example could be the Kalton–Peck  $Z_2$  space [19] which has no unconditional basis while containing an uncomplemented copy of  $\ell_2$  such that  $Z_2/\ell_2 = \ell_2$ . Recall that a short exact sequence of Banach spaces is a diagram

 $0 \longrightarrow Y \xrightarrow{i} X \xrightarrow{q} Z \longrightarrow 0 \tag{1}$ 

We show that, given a nonmetrizable compact space K having  $\omega$ -derived set empty,

there always exist nontrivial exact sequences  $0 \to c_0 \to E \to C(K) \to 0$ . This

partially solves a problem posed in several papers: Is  $Ext(C(K), c_0) \neq 0$  for K a

where Y, X and Z are Banach spaces and the arrows are operators in such a way that the kernel of each arrow coincides with the image of the preceding one. By the open mapping theorem i embeds Y as a closed

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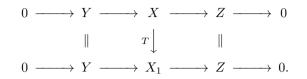


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subspace of X and Z is isomorphic to the quotient X/i(Y). Two exact sequences  $0 \to Y \to X \to Z \to 0$ and  $0 \to Y \to X_1 \to Z \to 0$  are said to be *equivalent* if there exists an operator  $T: X \to X_1$  making commutative the diagram



The sequence (1) is said to be trivial (or that it splits) if i(Y) is complemented in X (i.e., if it is equivalent to the sequence  $0 \to Y \to Y \oplus Z \to Z \to 0$ ). We write Ext(Z, Y) = 0 to indicate that every sequence of the form (1) splits. Summing all up, the general problem is to determine when there exist nontrivial twisted sums of Y and Z, or else, when  $\text{Ext}(Z, Y) \neq 0$ .

We pass now to consider the specific case in which both Y and Z are spaces of continuous functions on compact spaces. The twisting of separable C(K)-spaces was treated and, to a large extent, solved in [7]. Nevertheless, the problem of constructing nontrivial twisted sums of large C(K)-spaces is mostly unsolved. The following problem was posed and considered in [7,8,10]: Is it true that for every non-metrizable compact K one has  $\text{Ext}(C(K), c_0) \neq 0$ ? In this paper we prove that  $\text{Ext}(C(K), C(S)) \neq 0$  for any two compact spaces K, S with  $\omega$ -derived set empty and K nonmetrizable which, in particular, covers the case above for  $C(S) = c_0$ .

## 2. Main result, and its consequences

In what follows, the cardinal of a set  $\Gamma$  will be denoted  $|\Gamma|$ . We will write  $c_0(\Gamma)$  or  $c_0(|\Gamma|)$  depending if one needs to stress the set or only the cardinal.

We begin with a couple of observations: First of all, that if  $\Gamma$  is an uncountable set then  $\operatorname{Ext}(c_0(\Gamma), c_0) \neq 0$ . To explicitly construct an example it is enough to assume that  $|\Gamma| = \aleph_1$ , pick an isomorphic embedding  $\varphi : c_0(\aleph_1) \to \ell_{\infty}/c_0$  and consider the commutative diagram

where  $p_{\varphi}(x) = \varphi^{-1}p(x)$ . Indeed, to see that the lower sequence does not split, one only has to observe that in any commutative diagram

in which  $\varphi$  is an isomorphic embedding then also  $\varphi'$  is an isomorphic embedding, which implies that the quotient map q cannot be an isomorphism onto any non-separable subspace of  $c_0(\Gamma)$  since non-separable  $c_0(\Gamma')$  spaces are not subspaces of  $\ell_{\infty}$ . The lower sequence in diagram (2) can however split when  $\varphi$  is just an operator. Moreover, the results in [12] imply that in any exact sequence

$$0 \longrightarrow c_0 \longrightarrow X \xrightarrow{q} c_0(J) \longrightarrow 0$$

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