Topology and its Applications 198 (2016) 107–116

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Nonseparable $C(K)$-spaces can be twisted when K is a finite height compact ☆,☆☆

Jesús M.F. Castillo

Departamento de Matemáticas, Universidad de Extremadura, Avenida de Elvas, 06011, Badajoz, Spain

ARTICLE INFO

Article history:
Received 22 October 2013
Received in revised form 21 April 2015
Accepted 23 November 2015
Available online 7 December 2015

Keywords:
Spaces of continuous functions
Twisted sums

ABSTRACT

We show that, given a nonmetrizable compact space K having ω-derived set empty, there always exist nontrivial exact sequences $0 \to c_0 \to E \to C(K) \to 0$. This partially solves a problem posed in several papers: Is $\text{Ext}(C(K), c_0) \neq 0$ for K a nonmetrizable compact set?

1. Introduction

A general problem in Banach space theory is to determine, given two Banach spaces Y and Z, the existence and properties of Banach spaces X containing Y such that $X/Y = Z$. The space X is called a twisted sum of Y and Z. When only the trivial situation $X = Y \oplus Z$ is possible one obtains an interesting structure result asserting that every copy of Y in a space X such that $X/Y = Z$ must be complemented. When some $X \neq Y \oplus Z$ as above exists, one obtains a usually exotic and interesting space with unexpected properties; perhaps the perfect example could be the Kalton–Peck Z_2 space [19] which has no unconditional basis while containing an uncomplemented copy of ℓ_2 such that $Z_2/\ell_2 = \ell_2$. Recall that a short exact sequence of Banach spaces is a diagram

$$0 \longrightarrow Y \xrightarrow{\ i\ } X \xrightarrow{\ q\ } Z \longrightarrow 0 \tag{1}$$

where Y, X and Z are Banach spaces and the arrows are operators in such a way that the kernel of each arrow coincides with the image of the preceding one. By the open mapping theorem i embeds Y as a closed

☆ Thanks are due to a demanding referee who pushed the author to look for nicer arguments and, in the end, to produce a better paper. Additional thanks are due to Daniel Tausk who made a few accurate remarks about the content of the paper.
☆☆ This research was supported by project MTM2013-45643-C2-1-P, Spain.
E-mail address: castillo@unex.es.

http://dx.doi.org/10.1016/j.topol.2015.11.009
0166-8641/

subspace of X and Z is isomorphic to the quotient $X/i(Y)$. Two exact sequences $0 \to Y \to X \to Z \to 0$ and $0 \to Y \to X_1 \to Z \to 0$ are said to be *equivalent* if there exists an operator $T : X \longrightarrow X_1$ making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow T & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0. \end{array}$$

The sequence (1) is said to be trivial (or that it splits) if $i(Y)$ is complemented in X (i.e., if it is equivalent to the sequence $0 \to Y \to Y \oplus Z \to Z \to 0$). We write $\mathrm{Ext}(Z, Y) = 0$ to indicate that every sequence of the form (1) splits. Summing all up, the general problem is to determine when there exist nontrivial twisted sums of Y and Z, or else, when $\mathrm{Ext}(Z, Y) \neq 0$.

We pass now to consider the specific case in which both Y and Z are spaces of continuous functions on compact spaces. The twisting of separable $C(K)$-spaces was treated and, to a large extent, solved in [7]. Nevertheless, the problem of constructing nontrivial twisted sums of large $C(K)$-spaces is mostly unsolved. The following problem was posed and considered in [7,8,10]: Is it true that for every non-metrizable compact K one has $\mathrm{Ext}(C(K), c_0) \neq 0$? In this paper we prove that $\mathrm{Ext}(C(K), C(S)) \neq 0$ for any two compact spaces K, S with ω-derived set empty and K nonmetrizable which, in particular, covers the case above for $C(S) = c_0$.

2. Main result, and its consequences

In what follows, the cardinal of a set Γ will be denoted $|\Gamma|$. We will write $c_0(\Gamma)$ or $c_0(|\Gamma|)$ depending if one needs to stress the set or only the cardinal.

We begin with a couple of observations: First of all, that if Γ is an uncountable set then $\mathrm{Ext}(c_0(\Gamma), c_0) \neq 0$. To explicitly construct an example it is enough to assume that $|\Gamma| = \aleph_1$, pick an isomorphic embedding $\varphi : c_0(\aleph_1) \to \ell_\infty/c_0$ and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & \ell_\infty & \stackrel{p}{\longrightarrow} & \ell_\infty/c_0 & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow \varphi & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & p^{-1}\left(\varphi(c_0(\aleph_1))\right) & \stackrel{p_\varphi}{\longrightarrow} & c_0(\aleph_1) & \longrightarrow & 0 \end{array}$$

where $p_\varphi(x) = \varphi^{-1}p(x)$. Indeed, to see that the lower sequence does not split, one only has to observe that in any commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/c_0 & \longrightarrow & 0 \\ & & \| & & \uparrow \varphi' & & \uparrow \varphi & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & X & \stackrel{q}{\longrightarrow} & c_0(\Gamma) & \longrightarrow & 0 \end{array} \qquad (2)$$

in which φ is an isomorphic embedding then also φ' is an isomorphic embedding, which implies that the quotient map q cannot be an isomorphism onto any non-separable subspace of $c_0(\Gamma)$ since non-separable $c_0(\Gamma')$ spaces are not subspaces of ℓ_∞. The lower sequence in diagram (2) can however split when φ is just an operator. Moreover, the results in [12] imply that in any exact sequence

$$0 \longrightarrow c_0 \longrightarrow X \stackrel{q}{\longrightarrow} c_0(J) \longrightarrow 0$$