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Realizations of branched self-coverings of the 2-sphere

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A R T I C L E I N F O

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A B S T R A C T

For a degree-d branched self-covering of the 2-sphere, a notable combinatorial invariant is an integer partition of $2d-2$, consisting of the multiplicities of the critical points. A finer invariant is the so-called Hurwitz passport. The realization problem of Hurwitz passports remains largely open till today. In this article, we introduce two different types of finer invariants: a bipartite map and an incidence matrix. We then settle completely their realization problem by showing that a bipartite map, or a matrix, is realized by a branched covering if and only if it satisfies a certain balanced condition. A variant of the bipartite map approach was initiated by W. Thurston. Our results shed some new light to the Hurwitz passport problem.

1. Introduction

The main topic of this article is the study of branched covers from $\mathbb{S}^2$ to $\mathbb{S}^2$. Typical examples are meromorphic functions defined on the Riemann sphere. Our objective is to construct a finer invariant of these branched covers in order to bring a new point of view on the realization problems.

A map $\pi : \mathbb{S}^2 \to \mathbb{S}^2$ is called **a branched (or ramified) covering** of degree d, if there exists some finite subset $F \subset \mathbb{S}^2$ such that

- the restriction map $\pi : \mathbb{S}^2 \setminus \pi^{-1}(F) \to \mathbb{S}^2 \setminus F$ is a covering map of degree d;
- for each point $x \in F$, there is a neighborhood V of x, and a neighborhood U of each preimage y of x by π, such that the restricted map $\pi|_U : U \to V$ is equivalent, up to topological change of coordinates, to the map $z \mapsto z^k$ on the unit disc, for some integer $k > 0$.

This integer is uniquely determined for any point y of $\pi^{-1}(F)$ (and more generally for any point of $\mathbb{S}^2$), and it is called the **ramification number** of y. Notice that this number is equal to the number of preimages

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close to y, of a point close to x. In particular, this integer is equal to 1 if and only if π is locally a homeomorphism.

A point whose the ramification number is greater than 1 is called a **critical point**, and its image a **critical value**. Moreover, the sum of the ramification numbers of the preimages of any point in $\mathbb{S}^2$ is constant, equal to the degree of the branched covering. In other words, the ramification numbers of the preimages of any point x in $\mathbb{S}^2$ form an integer partition of d. And this partition is not the trivial partition $(1, \ldots, 1)$ if and only if x is a critical value.

Let π be a branched self-covering of degree d. Then we can associate to π two combinatorial invariants:

- the (unordered) list $l = (a_1, \ldots, a_m)$ of ramification numbers at the critical points, with $2 \leq a_i \leq d$ for every i;
- the **branch datum** $\mathcal{D} = [\Pi_1, \ldots, \Pi_n]$, with each Π_i a non-trivial integer partition of d, representing the collection of ramification numbers of the preimages of the i-th critical value.

Here, the integers m and n are respectively the number of critical points and the number of critical values of π. Notice also that the branch datum incorporates the information of all the other properties of π.

Our focus will be on the **realization problem** of these combinatorial properties. It can be expressed as follows:

Realization problem. Consider an integer $d \geq 2$. Given a list $l = (a_1, \ldots, a_m)$ of integers, with $2 \leq a_i \leq d$, or a list $\mathcal{D}$ of non-trivial integer partition of d, can it be realized by a branched covering of $\mathbb{S}^2$ of degree d?

This problem, in particular the realization problem of an abstract branch datum, is generally called the **Hurwitz problem** [8]. There exists an important necessary condition, called the **Riemann–Hurwitz condition** (we will reprove it along the way).

Let Π be an integer partition of d. We define the **weight** of Π, denoted $\nu(\Pi)$, by

$$\nu(\Pi) = d - card(\Pi).$$

And we define the **branched weight** (or total weight) of a list $\mathcal{D} = [\Pi_1, \ldots, \Pi_n]$ of such partitions by

$$\nu(\mathcal{D}) = \sum_{i=1}^{n} \nu(\Pi_i).$$

Riemann–Hurwitz condition. If a list $\mathcal{D} = [\Pi_1, \ldots, \Pi_n]$ of integer partitions of d is the branch datum of a branched covering of degree d, then

$$\nu(\mathcal{D}) = 2d - 2. \tag{1}$$

We can also rewrite this condition for the list of ramification numbers:

Riemann–Hurwitz condition. If a list of integers $l = (a_1, \ldots, a_m)$ is the list of ramification numbers at the critical points of a branched covering of degree d, then $2 \leq a_i \leq d$ for every i and $\sum_i (a_i - 1) = 2d - 2$.

Notice that the numbers m of critical points, and n of critical values of a branched covering of degree $d \geq 2$ satisfy $2 \leq n \leq m \leq 2d - 2$.

A list that satisfies the Riemann–Hurwitz condition will be called a **ramification distribution of degree** d, and a list $\mathcal{D}$ of integer partitions of d that satisfies the Riemann–Hurwitz condition is generally called a