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# Dynamical decomposition theorems of homeomorphisms with zero-dimensional sets of periodic points

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ABSTRACT

In this note, we study some dynamical decomposition theorems of spaces related to given homeomorphisms. First, we prove that if $f : X \to X$ is a homeomorphism of an $n$-dimensional separable metric space $X$ with zero-dimensional set of periodic points, then $X$ can be decomposed into a zero-dimensional bright space of $f$ except $n$ times and an $(n-1)$-dimensional dark space of $f$ except $n$ times, and also by use of dark spaces, we prove some decomposition theorems of $X$ related to dimension theory and dynamical systems. Finally, we study dynamical decompositions of continuum-wise expansive homeomorphisms.



## 1. Introduction

In this note, we assume that all spaces are separable metric spaces and dimension means the topological dimension dim. Also, let $\mathbb{N}$ and $\mathbb{Z}$ denote the set of natural numbers and the set of integers, respectively. If $A$ is a subset of a space $X$, then $\mathrm{cl}(A)$, $\mathrm{bd}(A)$ and $\mathrm{int}(A)$ denote the closure, the boundary and the interior of $A$ in $X$, respectively. For a collection $\mathcal{G}$ of subsets of $X$,

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$$\mathrm{ord}(\mathcal{G}) = \sup\{\mathrm{ord}_x(\mathcal{G})|\ x \in X\},$$

where $\mathrm{ord}_x(\mathcal{G})$ is the number of members of $\mathcal{G}$ which contains $x$.

In this note, we introduce new notions of 'bright spaces' and 'dark spaces' of homeomorphisms except $n$ times, and by use of the notions we prove some dynamical decomposition theorems of spaces related to given homeomorphisms. For a homeomorphism $f : X \to X$ of a space $X$ and $k \in \mathbb{N}$, let $P_k(f)$ denote the set of points of period $\leq k$. Also, $P(f)$ denotes the set of all periodic points of $f$. A subset $Z$ of $X$ is a *bright space* of $f$ except $n$ times ($n \in \{0\} \cup \mathbb{N}$) if for any $x \in X$,

$$|\{p \in \mathbb{Z}|\ f^p(x) \notin Z\}| \leq n,$$

where $|A|$ denotes the cardinality of a set $A$. Also we say that $L = X - Z$ is a *dark space* of $f$ except $n$ times. Note that for any $x \in X$, $|O_f(x) \cap L| \leq n$, where $O_f(x) = \{f^p(x)|\ p \in \mathbb{Z}\}$ denotes the orbit of $x$, and also note that $L \cap P(f) = \phi$. For a dark space $L$ of $f$ except $n$ times and $0 \leq j \leq n$, we put

$$A_f(L, j) = \{x \in X|\ |\{p \in \mathbb{Z}|\ f^p(x) \in L\}| = j\}\ (= \{x \in X|\ |O_f(x) \cap L| = j\}).$$

$A_f(L, j)$ denotes the set of all points $x \in X$ whose orbit $O_f(x)$ appears in $L$ just $j$ times. Note that $P(f) \subset A_f(L, 0)$ and $A_f(L, j)$ is $f$-invariant, i.e. $f(A_f(L, j)) = A_f(L, j)$ and $A_f(L, i) \cap A_f(L, j) = \phi$ if $i \neq j$. Hence we have the $f$-invariant decomposition related to the dark space $L$ as follows;

$$X = A_f(L, 0) \cup A_f(L, 1) \cup \cdots \cup A_f(L, n).$$

## 2. Dynamical decomposition theorems

It is well known that a space $X$ has at most dimension $n$ ($n \in \{0\} \cup \mathbb{N}$) (i.e. $\dim X \leq n$) if and only if $X$ can be represented as a union of $(n + 1)$ zero-dimensional subspaces of $X$ (see [2,12]). The following proposition may be known. For completeness, we give the proof.

**Proposition 2.1.** *Suppose that $X$ is a space with $\dim X = n$ ($< \infty$) and $f : X \to X$ is a homeomorphism. Then there exist $f$-invariant zero-dimensional dense $G_\delta$-sets $E_f(j)$ ($j = 0, 1, 2, \ldots, n$) of $X$ such that*

$$X = E_f(0) \cup E_f(1) \cup \cdots \cup E_f(n).$$

**Proof.** First, we prove the following claim (I); for each zero-dimensional $f$-invariant set $A$ of $X$, there is a zero-dimensional $f$-invariant $G_\delta$-set $\tilde{A}$ with $A \subset \tilde{A}$.

To prove the claim (I), choose a zero-dimensional $G_\delta$-set $A'$ with $A \subset A'$ (see [2]), i.e. $A' = \cap_{i \in \mathbb{N}} U_i$, where $U_i$ ($i \in \mathbb{N}$) is an open set of $X$. Then the set $\tilde{A} = \cap\{f^p(U_i)|\ i \in \mathbb{N}, p \in \mathbb{Z}\}$ is a desired $G_\delta$-set.

Since $X$ is separable, there is a countable dense set $D'$ of $X$ and we put $D = \cup\{f^p(D')|\ p \in \mathbb{Z}\}$. Then $D$ is countable, dense and $f$-invariant. We will show the following claim (II); there is a zero-dimensional $f$-invariant dense set $E_0$ of $X$ such that $D \subset E_0$ and

$$\dim\ [(X - E_0) \cup D] \leq n - 1 = \dim X - 1.$$

To prove the claim (II), choose a countable open base $\{U_i|\ i \in \mathbb{N}\}$ of $X$ such that $\mathrm{bd}(U_i) \cap D = \phi$ and $\dim \mathrm{bd}(U_i) \leq n - 1$ for each $i \in \mathbb{N}$ (see [2]). Put

$$L = \cup\{f^p(\mathrm{bd}(U_i))|\ i \in \mathbb{N}, p \in \mathbb{Z}\}.$$