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Weak pseudocompactness on spaces of continuous functions ☆

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A B S T R A C T

A space X is *weakly pseudocompact* if it is G_δ-dense in at least one of its compactifications. X has *property* D_Y if for every countable discrete and closed subset N of X, every function $f : N \to Y$ can be continuously extended to a function over all of X. *O-pseudocompleteness* is the pseudocompleteness property defined by J.C. Oxtoby [17], and *T-pseudocompleteness* is the pseudocompleteness property defined by A.R. Todd [24].

In this paper we analyze when a space of continuous functions $C_p(X,Y)$ is weakly pseudocompact where X and Y are such that $C_p(X,Y)$ is dense in Y^X. We prove: (1) For spaces X and Y such that X has property D_Y and Y is first countable weakly pseudocompact and not countably compact, the following conditions are equivalent: (i) $C_p(X,Y)$ is weakly pseudocompact, (ii) $C_p(X,Y)$ is O-pseudocomplete, and (iii) $C_p(X,Y)$ is T-pseudocomplete. (2) For a space X and a compact metrizable topological group G such that X has property D_G, the following statements are equivalent: (i) $C_p(X,G)$ is pseudocompact, (ii) $C_p(X,G)$ is weakly pseudocompact, (iii) $C_p(X,G)$ is T-pseudocomplete, and (iv) $C_p(X,G)$ is O-pseudocomplete. (3) For every space X, $C_p^*(X)$ and $C_p^*(X,\mathbb{Z})$ are not weakly pseudocompact. Throughout this study we also consider several completeness properties defined by topological games such as the Banach–Mazur game and the Choquet game.

1. Introduction

A well known result by E. Hewitt [12] states that a space X is pseudocompact if and only if X is G_δ-dense in βX. In [9], S. García-Ferreira and A. García-Máynez introduced the following concept: a topological space

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is *weakly pseudocompact* if it is G_δ-dense in one of its compactifications. So, every pseudocompact space is weakly pseudocompact. Also, they proved:

Theorem 1.1.

(1) *Every weakly pseudocompact space is a Baire space.*
(2) *If X is weakly pseudocompact and Lindelöf, then X is compact.*
(3) *Weak pseudocompactness is a productive property.*

Examples of weakly pseudocompact spaces are: (1) Every non-Lindelöf locally compact space; in particular, the discrete space $D(\kappa)$ of cardinality $\kappa > \omega$, and every Mrówka space $\Psi(\mathcal{A})$ with $|\mathcal{A}| > \omega$. (2) The completely metrizable hedgehog $J(\kappa)$ with $\kappa \neq \omega$ [8]. (3) For cardinal numbers $\omega < \kappa < \alpha$ such that $\kappa = \kappa^\omega$, the space $A_\kappa(\alpha) = D(\alpha) \cup \{o\}$ where $o \notin D(\alpha)$, and $B \subseteq A_\kappa(\alpha)$ is open if and only if either $B \subseteq D(\alpha)$ or $o \in B$ and $|D(\alpha) \setminus B| \leq \kappa$ [16].

On the other hand, every non-compact Lindelöf space is not weakly pseudocompact. Observe that ω^ω and $\mathbb{R}^\omega$ are not weakly pseudocompact because they are Lindelöf non-compact spaces. In [8], F.W. Eckertson asked the following fundamental question which is still open.

Question 1.2. Are the products ω^{ω_1} and $\mathbb{R}^{\omega_1}$ weakly pseudocompact spaces?

A more general problem to Question 1.2, the study of which could help understand the inner nature of Eckertson's problem, is to characterize spaces X and Y for which $C_p(X, Y)$ is weakly pseudocompact. Some contributions in the direction of solving this problem are made in [5] and [6]. Another significant class of spaces containing all pseudocompact spaces, contained in the class of Baire spaces and closed under arbitrary products is the class of pseudocomplete spaces introduced by J.C. Oxtoby in [17].

In this article we are going to study natural extensions of the classical results on completeness properties of $C_p(X)$ established by D.J. Lutzer and R.A. McCoy in [13], and further studied by V.V. Tkachuk in [23]. We will generalize these results to the wider class of function spaces of the form $C_p(X, Y)$ for Tychonoff spaces X and Y when $C_p(X, Y)$ is dense in Y^X, and for additional properties not previously considered, most notably the weak pseudocompactness. With this goal in mind, we study the pseudocompleteness defined by J.C. Oxtoby (O-pseudocompleteness), the pseudocompleteness defined by A.R. Todd (T-pseudocompleteness), and some type of completeness properties defined by topological games, and analyze the relations of these properties in $C_p(X, Y)$.

The work by Lutzer and McCoy [13] was primarily motivated to study the Baire property of $C_p(X)$. They determined necessary and sufficient conditions for which $C_p(X)$ is Baire when X is completely Hausdorff and the set $X' := \{x \in X : x \text{ is not isolated in } X\}$ is a discrete subset of X. The complete characterization of spaces X for which $C_p(X)$ is a Baire space was given simultaneously by E. van Douwen in a private communication (see [14]) and E.G. Pytkeev [18]. The characterization of the Baire property for spaces of the form $C_p(X, Y)$ is a pending task.

On the other hand, observe that if Y is not pseudocompact, $C_p(X, Y)$ is not pseudocompact. Indeed, if $h : Y \to \mathbb{R}$ is continuous and not bounded, then the function $\mu : C_p(X) \to \mathbb{R}$ defined by $\mu = h \circ \phi$, where $\phi : C_p(X) \to Y$ is $\phi(f) = f(x_0)$ for a beforehand fixed point $x_0 \in X$, is continuous and not bounded.

In Section 3 we analyze some generalizations of C-embeddedness and discreteness properties which will help us study weak pseudocompactness in function spaces with the pointwise convergent topology. Mainly, we study u_Y-discrete spaces; that is, spaces X where every countable subset N is discrete and every continuous function $f : N \to Y$ can be continuously extended to all of X. In Section 4 we prove that $C_p(X, Y)$ is weakly pseudocompact if Y is weakly pseudocompact and X is u_Y-discrete. Section 5 is devoted to some technical results related to weakly α-favorable spaces over which we will construct our main results. In Section 6 we