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A generalized central sets theorem and applications

Dev Phulara [1]

Department of Mathematics, Howard University, Washington, DC 20059, United States

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ABSTRACT

The central sets theorem originally proven by H. Furstenberg is a powerful result which is applicable to derive many combinatorial conclusions. Furstenberg's original theorem applied to $\mathbb{N}$ and finitely many sequences in $\mathbb{Z}$. Some strengthenings of this theorem have been derived first by V. Bergelson and N. Hindman in 1990. Later in 2008, D. De, N. Hindman, and D. Strauss proved a stronger version of the central sets theorem for arbitrary semigroups S which applied to all sequences in S. We provide here a generalization of the stronger version and some applications of this new generalization.

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1. Introduction

We start with a very brief history of the central sets theorem. Interested readers can find detailed history in the chapter notes after Chapter 14 in [7]. In fact [7] is the principal resource for this paper so the readers are requested to see [7] for any unfamiliar notions. The study of the central sets in $\mathbb{N}$ began with the following theorem by Furstenberg. (For any set A, $\mathcal{P}_f(A)$ denotes the set of all finite nonempty subsets of A.)

Theorem 1.1. *Let $l \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, l\}$, let $\langle y_{i,n} \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{Z}$. Let C be a central subset of $\mathbb{N}$. Then there exist sequences $\langle a_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) *for all n, $\max H_n < \min H_{n+1}$ and*
(2) *for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $i \in \{1, 2, \ldots, l\}$, $\sum_{n \in F}(a_n + \sum_{t \in H_n} y_{i,t}) \in C$.*

One can find the proof of this theorem in [4, Proposition 8.21]. This theorem is such a powerful result that not only does it provide a simultaneous generalization of two celebrated theorems (Hindman's theorem

E-mail address: phulara@comcast.net.
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and van der Waerden's theorem) but also their stronger forms. For example, it is an easy consequence that for any finite partition of $\mathbb{N}$ and for any given sequence $\langle x_n \rangle_{n=1}^{\infty}$, there exists a cell of that partition which contains an arbitrarily long arithmetic progression whose increment can be chosen from $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$. (See [7, Corollary 14.13].)

Despite being a combinatorial result, a combinatorial proof of the central sets theorem has not been found yet. Furstenberg's original proof utilized notions in topological dynamics, whereas V. Bergelson and N. Hindman used a different but equivalent definition of central sets using the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ and proved the following extension of Theorem 1.1.

Theorem 1.2. *Let $(S, +)$ be a commutative semigroup. Let $l \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, l\}$, let $\langle y_{i,n} \rangle_{n=1}^{\infty}$ be a sequence in S. Let C be a central subset of S. Then there exist sequences $\langle a_n \rangle_{n=1}^{\infty}$ in S and $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) *for all n, $\max H_n < \min H_{n+1}$ and*
(2) *for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $f : F \to \{1, 2, \ldots, l\}$, $\sum_{n\in F}(a_n + \sum_{t\in H_n} y_{f(i),t}) \in C$.*

Proof. See [1, Corollary 2.10]. □

The version of the central sets theorem we are interested in is due to D. De, N. Hindman, and D. Strauss which states the following. (For any two sets X and Y, ${}^{X}Y$ denotes the set of all functions with domain X and range contained in Y. In particular ${}^{\mathbb{N}}S$ is the set of all sequences in S.)

Theorem 1.3. *Let $(S, +)$ be a commutative semigroup and let C be a central subset of S. There exist functions $\alpha : \mathcal{P}_f({}^{\mathbb{N}}S) \to S$ and $H : \mathcal{P}_f({}^{\mathbb{N}}S) \to \mathcal{P}_f(\mathbb{N})$ such that*

(1) *if $F, G \in \mathcal{P}_f({}^{\mathbb{N}}S)$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and*
(2) *if $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f({}^{\mathbb{N}}S)$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $\langle y_{i,n} \rangle_{n=1}^{\infty} \in G_i$, then*

$$\sum_{i=1}^{m} \Big(\alpha(G_i) + \sum_{t\in H(G_i)} y_{i,t}\Big) \in C.$$

The proof of this theorem can be found in [2, Theorem 2.2] or in [7, Theorem 14.8.4]. Let's note that the proof of our main result in the subsequent section is highly motivated by that proof. It may appear that Theorem 1.2 does not generalize Theorem 1.1 because in Theorem 1.1 the sequences are allowed to come from $\mathbb{Z}$. As described in [2], it is a fact that any set central in $(\mathbb{N}, +)$ is also central in $(\mathbb{Z}, +)$. So Theorem 1.2 does generalize the original central sets theorem. And one can show Theorem 1.2 as a corollary of Theorem 1.3 by an argument similar to that we use in Corollary 2.8.

We use the definition of central sets in terms of Stone–Čech compactification. The equivalence between this and Furstenberg's original definition of the central sets can be found in [9] or in [7, Theorem 19.27]. We now briefly describe the algebraic structure of the Stone–Čech compactification of a discrete semigroup S. For detailed information see [7].

For any discrete semigroup S, we define the *Stone–Čech compactification* βS as the set of all ultrafilters on S and we identify the principal ultrafilters with the points of S. For p and q in βS, we define $p \cdot q = \{A \subseteq S : \{x \in S : x^{-1}A \in q\} \in p\}$. That is $A \subseteq S$ is in $p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : xy \in A\}$. For $A \subseteq S$, we define $\overline{A} = \{p \in \beta S : A \in p\}$. It turns out that the collection $\{\overline{A} : A \subseteq S\}$ forms a basis for a compact Hausdorff topology on βS; for each $s \in S$, the function $\lambda_s : \beta S \to \beta S$ defined by $\lambda_s(q) = s \cdot q$ is continuous; and for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$

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