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Milnor invariants and edge-homotopy classification of clover links

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We define Milnor numbers for clover links and show that for a clover link, if Milnor numbers of length $\leq k$ vanish, then Milnor numbers of length $\leq 2k+1$ are well-defined. Moreover we prove that two clover links whose Milnor numbers of length $\leq k$ vanish are equivalent up to edge-homotopy and C_{2k+1}-equivalence if and only if those Milnor numbers of length $\leq 2k+1$ are equal. In particular, we give an edge-homotopy classification of 3-clover links by their Milnor numbers of length ≤ 3.

1. Introduction

In 1954, J. Milnor [9] introduced the notion of *link-homotopy*, which is a weaker equivalence relation than link type and generated by crossing changes on the same component. He defined *Milnor invariants* which are given as follows (see [9,10] and Subsection 2.3 for detailed definitions). Let L be an oriented ordered n-component link in the 3-sphere S^3. The *Milnor number* $\mu_L(I)$ is an integer determined by a finite sequence I of numbers in $\{1, 2, \ldots, n\}$. The *Milnor invariant* $\overline{\mu}_L(I)$ is the residue class of $\mu_L(J)$ modulo the greatest common divisor of the $\mu_L(J)'s$, where J is obtained from a proper subsequence of I by permuting cyclicly. The length of I is called the *length* of $\overline{\mu}_L(I)$ and denoted by $|I|$.

Milnor [9] proved that Milnor invariants for non-repeated sequences are link-homotopy invariants and gave a link-homotopy classification of 2- and 3-component links by these invariants. In 1988, J.P. Levine [8] gave a link-homotopy classification of 4-component links. In 1990, N. Habegger and X.S. Lin [3] gave an algorithm which determines if two links with arbitrarily many components are link-homotopic. In [3], they defined Milnor numbers for string links which are similarly defined as links and proved that Milnor numbers are integer invariants for these objects. Moreover they showed that Milnor numbers for non-repeated sequences

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give a link-homotopy classification of string links with arbitrarily many components. We remark that Milnor numbers are complete link-homotopy invariants for string links, but Milnor invariants are not strong enough to classify links up to link-homotopy.

Let C_n be a graph consisting of n loops, such that the loops are connected by edges to one common vertex. An embedded C_n in S^3 is called an *n-clover link* in S^3 [7]. Given an n-clover link c, there is a well-defined associated link l_c, given by the disjoint union of the loops of c. There is also a natural procedure to construct an associated string link $\gamma(c)$. We define the *Milnor number* μ_c to be the Milnor number $\mu_{\gamma(c)}$. We remark that there are infinitely many choices of $\gamma(c)$ for c, and hence that, in general, μ_c is not an invariant of c. But under a certain condition, μ_c is well-defined as stated in the theorem below.

Theorem (Theorem 2.2). *For an n-clover link c, if $\overline{\mu}_{l_c}(J) = 0$ for any sequence with $|J| \leq k$, then $\mu_c(I)$ is well-defined for any sequence with $|I| \leq 2k+1$.*

In 1988, Levine [7] already defined Milnor numbers for *flat vertex* clover links and proved the same result as the theorem above, where a flat vertex spatial graph is a spatial graph Γ such that for each vertex v of Γ, there exist a neighborhood B_v of v and a small flat plane P_v such that $\Gamma \cap B_v \subset P_v$ [13]. In this paper, we do not assume that clover links are flat vertex ones.

The *edge-homotopy* is an equivalence relation on spatial graphs generated by crossing changes on the same edge. This equivalence relation was introduced by K. Taniyama [12] as a generalization of link-homotopy. The *C_k-equivalence*, defined by K. Habiro [4], is an equivalence relation generated by a kind of higher order crossing change. The *(edge-homotopy+C_k)-equivalence* is the equivalence relation generated by combining edge-homotopy with C_k-equivalence.

Theorem (Theorem 3.3). *Let c and c' be two n-clover links. If $\overline{\mu}_{l_c}(J) = \overline{\mu}_{l_{c'}}(J) = 0$ for any sequence with $|J| \leq k$, then c and c' are (edge-homotopy+C_{2k+1})-equivalent if and only if $\mu_c(I) = \mu_{c'}(I)$ for any non-repeated sequence with $|I| \leq 2k+1$.*

As a consequence, we obtain a characterization of edge-homotopy for clover links in terms of Milnor numbers, again under a vanishing assumption (except for the case of 3-clover links).

Proposition (Corollary 3.4). *Let c and c' be two n-clover links. If $\overline{\mu}_{l_c}(J) = \overline{\mu}_{l_{c'}}(J) = 0$ for any sequence with $|J| \leq n/2$, then c and c' are edge-homotopic if and only if $\mu_c(I) = \mu_{c'}(I)$ for any non-repeated sequence with $|I| \leq n$.*

Since Milnor invariants of length 1 always vanish, the proposition above implies the following.

Corollary (Corollary 3.5). *Two 3-clover links c and c' are edge-homotopic if and only if $\mu_c(I) = \mu_{c'}(I)$ for any non-repeated sequence with $|I| \leq 3$.*

Remark 1.1. Let c be a clover link. By definition of Milnor numbers for c in Subsection 2.3, we note that $\mu_c(I) = 0$ for any non-repeated sequence I if and only if $\overline{\mu}_{l_c}(I) = 0$. We also note that all Milnor numbers for a trivial clover link vanish, where a clover link is *trivial* if there is an embedded plane in S^3 which contains the clover link. It is shown by Milnor [9] that a link is link-homotopic to a trivial link if and only if all Milnor invariants for non-repeated sequences vanish. Hence, by the proposition above, we have that c is edge-homotopic to a trivial clover link if and only if l_c is link-homotopic to a trivial link.

2. Milnor numbers for clover links

In this section, we will define Milnor numbers for clover links.