Topology and its Applications 194 (2015) 4–21

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Random knots using Chebyshev billiard table diagrams

Moshe Cohen [1], Sunder Ram Krishnan [*,2]

Department of Electrical Engineering, Technion, Haifa 32000, Israel

A R T I C L E I N F O

Article history:
Received 16 June 2015
Received in revised form 28 July 2015
Accepted 30 July 2015
Available online 11 August 2015

MSC:
57M25
57M27
05C80
60C05
60B99

Keywords:
Two-bridge
Conway form

A B S T R A C T

We use the Chebyshev knot diagram model of Koseleff and Pecker in order to introduce a random knot diagram model by assigning the crossings to be positive or negative uniformly at random. We give a formula for the probability of choosing a knot at random among all knots with bridge index at most 2. Restricted to this class, we define internal and external reduction moves that decrease the number of crossings of the diagram. We make calculations based on our formula, showing the numerics in graphs and providing evidence for our conjecture that the probability of any knot appearing in this model decays to zero as the number of crossings goes to infinity.

1. Introduction

It is still unknown whether the Jones polynomial detects the unknot [22, Problem 1], that is, whether the Jones polynomial of some non-trivial knot is also equal to 1, the Jones polynomial of the unknot. One might believe this to be true because Khovanov homology, a knot invariant of bigraded abelian groups whose graded Euler characteristic gives the Jones polynomial, has been shown by Kronheimer and Mrowka to detect the unknot [29]. Dasbach and Hougardy [12] found no counterexample in all knots up to seventeen crossings, including over two million prime knots, and this was extended to eighteen crossings by Yamada [42]. Rolfsen with Anstee and Przytycki [4] and with Jones [24] sought a counterexample via mutations. Bigelow [6] outlined an algorithm to find a counterexample among four-strand braids.

* Corresponding author.
E-mail addresses: mcohen@tx.technion.ac.il (M. Cohen), eeksunderram@gmail.com (S.R. Krishnan).

[1] The first author was supported in part by the funding from the European Research Council under the European Union's Seventh Framework Programme, Grant FP7-ICT-318493-STREP.
[2] The second author was supported in part by the funding from the European Research Council under Understanding Random Systems via Algebraic Topology (URSAT), ERC Advanced Grant ERC-2012-ADG_20120216.

http://dx.doi.org/10.1016/j.topol.2015.07.018
0166-8641/

We propose using the probabilistic method, pioneered by Erdös in the 1940's, as a tool to investigate data surrounding this question. This method has been successful in studying "typical" large "random" graphs, like the Erdös–Rényi model, and in showing that certain graph theoretic properties hold with certain probabilities for these large graphs. However, it has been only minimally exploited in topology in general, with some notable exceptions: the study of random 3-manifolds by Dunfield with W. Thurston [17] and D. Thurston [16]; random walks on the mapping class group in a recent work by Ito [21]; and the Linial–Meshulam model [33] for random simplicial 2-complexes and its recent generalization by Costa and Farber [9] to k-complexes. Also see survey papers on the topology of random complexes by Kahle [25] and Bobrowski and Kahle [7].

Ultimately, we would like to compare the probability of obtaining the unknot with that of obtaining unit Jones polynomial, but for this we need a useful notion of a space of knots. Various models for random knots from the literature, appearing more frequently since an AMS Special Session in Vancouver in 1993, generally take one of two forms: the physical knotting of a random walk, as in [1,3,8,13–15], or a random walk on the Cayley graph of the braid group, as in [35,37,38,41].

Unfortunately, these models do not make common knot theoretic computations accessible. For this reason, we are interested in constructing a random model based on a particular knot diagram. The first paper on this topic appeared in 2014 by Even-Zohar, Hass, Linial, and Nowik [19] using the übercrossing and petal projections of Adams et al. [2]. Forthcoming work of Dunfield, Obeidin, et al. [18] is expected to be on the topic of random knot diagrams as well.

In this paper, we apply the probabilistic method to a model developed by Koseleff and Pecker [30–32] that uses Chebyshev polynomials of the first kind to parametrize a knotted curve. These so-called *Chebyshev knots* are analogues of the more established Lissajous knots studied by Jones and Przytycki [23]. However, Koseleff and Pecker show in [31] that all knots are Chebyshev, which is not the case for Lissajous knots (see for example [5]).

The goal of this paper is to establish a framework that could be used in future work to find evidence surrounding the question stated above. However, the present scope will be restricted to the class of knots whose bridge index is at most 2, and it is already known that the Jones polynomial detects the unknot among this class. This is because 2-bridge knots are alternating [20] and Jones polynomials of alternating knots have spans (or degrees, after some normalization) equal to four times the crossing number of the alternating diagram. This achieves the minimal crossing number by Tait's Conjecture, proven by Kauffman [26,27], Murasugi [36], and Thistlethwaite [40].

One reason in particular why this model is of interest for the question above is that the Jones polynomial follows some nice recursions, as shown with formulas appearing in recent work by the first author [10].

Results. Let $T(3, n+1)$ be the billiard table diagram of a Chebyshev knot with bridge index at most 2 for $3 \nmid (n+1)$ and with crossings determined as below. Consider its projection to a 4-valent graph as in Fig. 3, where its vertices are ordered from left to right. Choose uniformly at random an element of $\{+,-\}^n$ to assign positive and negative crossings to the vertices of the projection. In the result below, we use $\xi_i = \xi_i(K) := x_i^{(i)}(K) = x_i^{(i)}$ that depend on a fixed knot K, as in Definition 3.1.

Main Theorem 4.2 (1). *The probability of a given knot K appearing in $T(3, n+1)$ is*

$$\frac{1}{2^n}\left[\xi_{n+3} + \sum_{i=0}^{n-6}\left[\binom{n-5}{i+1} + 4\binom{n-5}{i}\right]\xi_{n-3i} + 4\xi_{-2n+15}\right].$$

The second and third parts of this Main Theorem 4.2 give formulas for the variables ξ_i when i is positive and nonpositive, respectively.