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On convergent sequences in the Stone space of one Boolean algebra ☆

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ABSTRACT

We consider the Stone space of the Boolean algebra, constructed by M.G. Bell (see Example 2.1 in [2]), which is the compactification, BN, of the countable discrete space N. Here we consider convergent sequences in $BN \setminus N$.

We prove that if a point $x \in BN \setminus N$ is the limit of some sequence $\{s_n : n \in \omega\}$ from N, or a point $x \in [A] \setminus A$, where $A \subseteq N$ is a strict anti-chain in N, then there is a sequence in $BN \setminus N$, that converges to x.

We also prove that if A is a countable discrete set of u-points in $BN \setminus N$ and $x \in [A] \setminus A$, then x is not the limit of any sequence of points of $BN \setminus N$.

1. Introduction

We consider the Stone space of the Boolean algebra, constructed by M.G. Bell (see Example 2.1 in [2]). This space, BN, is the compactification of the countable discrete space N with ccc nonseparable remainder.

In [6] we proved that if $A \subseteq N$ is a infinite chain, then $|[A] \setminus A| = 1$, i.e. A is a convergent sequence [6, Theorem 3.9] and if $A \subseteq N$ is a strict anti-chain (see the definition in 2), then $[A]$ is homeomorphic to $\beta\omega$ [6, Lemma 3.5].

In [5] we proved that if $A \subseteq N$ is such that $|[A] \setminus A| = 1$, then $A \setminus K$ is a chain for some finite $K \subseteq A$ [5, Theorem 3.1].

We will consider the problem of existence of convergent sequences in the remainder $BN \setminus N$ of the space BN.

In [9] we proved that there is a convergent sequence in $BN \setminus N$ [9, Example 1].

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Here we prove that if $x \in BN \setminus N$ is such that x is the limit of some sequence $\{s_n : n \in \omega\} \subseteq N$, then there is a sequence $\{y_k : k \in \omega\} \subseteq BN \setminus N$, that converges to x (Theorem 3.5), and if $A \subseteq N$ is a strict anti-chain in N and a point $x \in [A] \setminus A$, then there is a sequence in $BN \setminus N$, that converges to x (Theorem 3.6).

We also prove that if A is a countable discrete set of u-points in $BN \setminus N$ and $x \in [A] \setminus A$, then x is not a limit of any sequence of points of $BN \setminus N$ (Theorem 3.15).

From these theorems it follows:

- there is a copy F of $\beta\omega \setminus \omega$ in the space $BN \setminus N$ such that every point $x \in F$ is the limit of some sequence of points of $BN \setminus N$ (Corollary 3.7);
- there is a copy F of $\beta\omega \setminus \omega$ in the space $BN \setminus N$ such that every point $x \in F$ is not the limit of any sequence of points of $BN \setminus N$ (Corollary 3.16);
- there is a point $x \in BN \setminus N$, sequences $\{x_n : n \in \omega\}$ and $\{y_m : m \in \omega\}$ in $BN \setminus N$ such that $\{x_n : n \in \omega\}$ converges to x, the closure $[\{y_m : m \in \omega\}]$ is homeomorphic to $\beta\omega$ and $\{x\} = [\{x_n : n \in \omega\}] \cap [\{y_m : m \in \omega\}]$ (Corollary 3.8).

2. Preliminaries

Definitions and notions used in the paper can be found in [1,3].

By $[A]$ we denote the closure of a set A.

For a mapping $f: X \to Y$ and a set $A \subseteq X$ we denote by $f|_A$ the restriction of f on A.

The construction of Bell's space [2] follows.

Let

$$P = \{f \in \omega^\omega : 0 \le f(n) \le n+1 \text{ for all } n \in \omega\}$$

and

$$N = \{f|_n : f \in P, n \in \omega\}$$

Define

$$T = \{\pi \in N^\omega : \operatorname{dom} \pi(n) = n+1 \text{ for all } n \in \omega\}$$

For every $s \in N$ let

$$C_s = \{t \in N : t|_{\operatorname{dom} s} = s\}.$$

The mapping $\pi \in T$ generates the set

$$C_\pi = \cup\{C_{\pi(n)} : n \in \omega\}.$$

Let B be the Boolean algebra, generated by

$$B' = \{C_\pi : \pi \in T\}$$

in the power set of N.

Note that $\{\{s\} : s \in N\} \cup \{C_s : s \in N\} \subseteq B$.

Denote by BN the Stone space of B. We are identifying each $s \in N$ with the fixed ultrafilter $\xi_s \in BN$ such that $\{s\} \in \xi_s$.