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A pseudometric invariant under similarities in the hyperspace of non-degenerated compact convex sets of $\mathbb{R}^n$

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ABSTRACT

In this work we define a new pseudometric in $\mathcal{K}^n_*$, the hyperspace of all non-degenerated compact convex sets of $\mathbb{R}^n$, which is invariant under similarities. We will prove that the quotient space generated by this pseudometric (which is the orbit space generated by the natural action of the group of similarities on $\mathcal{K}^n_*$) is homeomorphic to the Banach–Mazur compactum $BM(n)$, while $\mathcal{K}^n_*$ is homeomorphic to the topological product $Q \times \mathbb{R}^{n+1}$, where Q stands for the Hilbert cube. Finally we will show some consequences in convex geometry, namely, we measure how much two convex bodies differ (by means of our new pseudometric) in terms of some classical functionals.

1. Introduction

The most common way to measure the distance between two non-empty closed subsets of a metric space is by means of the well known Hausdorff distance. Namely, if A and B are compact subsets of the metric space (X, d), the Hausdorff distance between A and B is defined by the rule

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$$d_H(A,B) = \max\left\{\sup_{a\in A}\{d(a,B)\},\ \sup_{b\in B}\{d(b,A)\}\right\},$$

where $d(a,B) = \inf\{d(a,b) \mid b \in B\}$. However, this distance does not tell us much information about how much A and B are geometrically alike. In that sense, there are some other ways to measure the distance between (classes of) closed sets. For example, the Gromov–Hausdorff distance d_{GH} between two compact metric spaces X and Y is defined as the infimum of all Hausdorff distances $d_H(j(X), i(Y))$, where $j : X \to Z$ and $i : Y \to Z$ are isometric embeddings into a common metric space Z, d_H is the Hausdorff distance determined by the metric in Z and the infimum is taken over all possible Z, j, and i (see e.g. [13]). If A and B are isometric compact spaces, the Gromov–Hausdorff distance between them is always zero (and reciprocally). In general, d_{GH} measures how far two metric spaces are from being isometric, but this is not very helpful when we want to measure the difference between two compact spaces with respect to other geometric qualities rather than the isometries.

Take into account the following more particular situation. Suppose that we have a continuous action of a topological group G on a metric space (X,d). This action induces a continuous action on the hyperspace $(\mathcal{C}(X), d_H)$ of all non-empty compact subsets of X via the formula:

$$(g,A) \longmapsto gA = \{ga \mid a \in A\}, \quad A \in \mathcal{C}(X),\ g \in G \tag{1.1}$$

(continuity of this action will be prove later in Lemma 3.2). In this case we are interested in finding a useful pseudometric ρ in $\mathcal{C}(X)$ such that

$$\rho(A,B) = 0 \quad \text{if and only if} \quad A = gB \quad \text{for some} \quad g \in G. \tag{1.2}$$

$$\rho(gA, hB) = \rho(A,B) \quad \text{for every} \quad g,h \in G. \tag{1.3}$$

One way to achieve this consists in finding a metric δ in the orbit space $\mathcal{C}(X)/G = \{G(A) \mid A \in \mathcal{C}(X)\}$, where $G(A) = \{gA \mid g \in G\}$ denotes the G-orbit of A. In this case, the function ρ defined by the rule $\rho(A,B) = \delta(G(A), G(B))$ will satisfy the desired conditions.

Another well-known example which is closer to our interest is the Banach–Mazur distance between convex sets. Consider the set $\mathcal{K}_0^n$ of all convex bodies of $\mathbb{R}^n$ (i.e., all compact and convex subsets of $\mathbb{R}^n$ with non-empty interior) equipped with the natural action of the group $\mathrm{Aff}(n)$ of all invertible affine transformations of $\mathbb{R}^n$. The extended Banach–Mazur distance (or Minkowski distance) in $\mathcal{K}_0^n$ is defined as

$$d_{BM}(A,B) = \inf\{\alpha > 1 \mid \exists\, g \in \mathrm{Aff}(n),\ x \in \mathbb{R}^n \text{ such that } A \subset gB \subset \alpha A + x\}$$

It is well known that the Banach–Mazur distance satisfies the following properties

i) $d_{BM}(A,B) \geq 1$ and $d_{BM}(A,B) = 1$ iff $A = gB$ for some $g \in \mathrm{Aff}(n)$,
ii) $d_{BM}(A,B) = d_{BM}(B,A)$,
iii) $d_{BM}(A,B) \leq d_{BM}(A,C) \cdot d_{BM}(C,B)$.

Therefore, by taking the logarithm of d_{BM} we can define a pseudometric in $\mathcal{K}_0^n$ which measures how far two convex bodies are from belonging to the same $\mathrm{Aff}(n)$-orbit. If we equip the orbit space $\mathcal{K}_0^n/\mathrm{Aff}(n)$ with the metric induced by $\ln(d_{BM})$ (which in fact determines the quotient topology generated by the Hausdorff distance topology of $\mathcal{K}_0^n$), we obtain a compact metric space known as the Banach–Mazur compactum $BM(n)$. Let us recall that originally, the Banach–Mazur compactum $BM(n)$ was defined as the set of isometry classes of n-dimensional Banach spaces topologized by the original Banach–Mazur distance,