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Deformation classification of typical configurations of 7 points in the real projective plane

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A configuration of 7 points in $\mathbb{RP}^2$ is called *typical* if it has no collinear triples and no coconic sextuples of points. We show that there exist 14 deformation classes of such configurations. This yields classification of real Aronhold sets.

> *"This is one of the ways in which the magical number seven has persecuted me."*
>
> George A. Miller, *The magical number seven, plus or minus two: some limits of our capacity for processing information*

1. Introduction

1.1. Simple configurations of $n \leqslant 7$ points

Projective configurations of points on the plane is a classical subject in algebraic geometry and its history in the context of linear systems of curves can be traced back to 18th century (G. Cramer, L. Euler, etc.).

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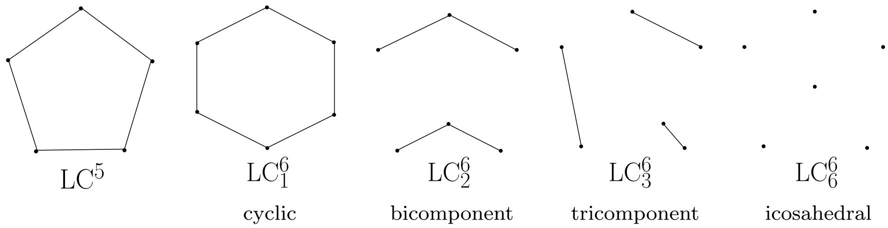

Fig. 1. Adjacency graphs $\Gamma_{\mathcal{P}}$ of 5- and 6-configurations (cyclic, bicomponent, tricomponent and icosahedral).

In modern times, projective configurations are studied both from algebro-geometric viewpoint (Geometric Invariant Theory, Hilbert schemes, del Pezzo surfaces), and from combinatorial geometric viewpoint (Matroid Theory). In the latter approach just linear phenomena are essential, and in particular, a generic object of consideration is a *simple n-configuration*, that is a set of n points in $\mathbb{RP}^2$ in which no triple of points is collinear. The dual object is *a simple n-arrangement*, that is a set of n real lines containing no concurrent triples.

A combinatorial characterization of a simple n-arrangement is its *oriented matroid*, which is roughly speaking a description of the mutual position of its partition polygons. For simple n-configurations it is essentially a description how do the plane lines separate the configuration points (see [1] for precise definitions). Such a combinatorial description was given for simple n-arrangements with $n \leqslant 7$ in [2] and [12]. In the beginning of 1980s N. Mnëv proved his universality theorem and in particular, constructed examples of combinatorially equivalent simple configurations which cannot be connected by a deformation. His initial example with $n \geqslant 19$ was improved by P. Suvorov (1988) to $n = 14$, and recently (2013) by Y. Tsukamoto to $n = 13$. Mnëv's work motivated the first author to verify in [7] (see also [8]) that for $n \leqslant 7$ the deformation classification still coincides with the combinatorial one, or in the other words, to prove connectedness of the realization spaces of the corresponding oriented matroids. One of applications of this in Low-dimensional topology was found in [10], via the link to the geometry of Campedelli surfaces.

As n grows, a combinatorial classification of simple n-configurations becomes a task for computer enumeration: there exist 135 combinatorial types of simple 8-arrangements (R. Canham, E. Halsey, 1971, J. Goodman and R. Pollack, 1980) and 4381 types of simple 9-arrangements (J. Richter-Gebert, G. Gonzales-Springer and G. Laffaille, 1989). The classification includes analysis of arrangements of *pseudolines* (oriented matroids of rank 3), their *stretchability* (realizability by lines) and analysis of connectedness of the realization space of a matroid that gives a deformation classification (see [1, Ch. 8] for more details).

In what follows, we need only the following summary of the deformation classification of simple n-configurations for $n \leqslant 7$. For $n = 5$ it is trivial: simple 5-configurations form a single deformation component, denoted by LC^5. This is because the points of such a configuration lie on a non-singular conic. For $n = 6$ there are 4 deformation classes shown on Fig. 1. On this figure, we sketched configurations $\mathcal{P}$ together with some edges (line segments) joining pairs of points, $p, q \in \mathcal{P}$. Namely, we sketch such an edge if and only if it is not crossed by any of the lines connecting pairs of the remaining $n - 2$ points of $\mathcal{P}$. The graph, $\Gamma_{\mathcal{P}}$, that we obtain for a given configuration $\mathcal{P}$ will be called the *adjacency graph* of $\mathcal{P}$ (in the context of the oriented matroids, there is a similar notion of *inseparability graphs*). For $n = 6$, the number of its connected components, 1, 2, 3, or 6, characterizes $\mathcal{P}$ up to deformation. The deformation classes of 6-configurations with i components are denoted LC^6_i, $i = 1, 2, 3, 6$, and the configurations of these four classes are called respectively *cyclic, bicomponent, tricomponent*, and *icosahedral* 6-configurations.

Given a simple 7-configuration $\mathcal{P}$, we label a point $p \in \mathcal{P}$ with an index $\delta = \delta(p) \in \{1, 2, 3, 6\}$ if $\mathcal{P} \smallsetminus \{p\} \in \mathrm{LC}^6_\delta$. Count of the labels gives a quadruple $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_6)$, where $\sigma_k \geqslant 0$ is the number of points $p \in \mathcal{P}$ with $\delta(p) = k$. We call $\sigma = \sigma(\mathcal{P})$ the *derivative code* of $\mathcal{P}$. There exist 11 deformation classes of simple 7-configurations that are shown on Fig. 2, together with their adjacency graphs and labels $\delta(p)$.