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Complex cobordism of quasitoric orbifolds

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We construct orbifolds with quasitoric boundary and show that they have stable almost complex structure. We show that a quasitoric orbifold is complex cobordant to finite disjoint copies of complex orbifold projective spaces. Finally some computations in the complex cobordism ring for manifolds are given.

1. Introduction

Cobordism was explicitly introduced by Lev Pontryagin in his seminal paper [15]. In [20] Thom showed that the cobordism groups could be computed by results of homotopy theory using the Thom complex construction. Later, Atiyah [2] showed that complex cobordism is a generalized cohomology theory. In Section 1 of [17], Quillen discussed geometric interpretation of complex cobordism rings. Following his definition, we define the complex orbifold (co)bordism groups and rings for the category of stable almost complex orbifolds. It seems that complex cobordism for orbifolds did not appear in the literature until now. However oriented cobordism of orbifolds is studied in [8] and [9].

In the pioneering paper [7], Davis and Januskiewicz introduced the topological counterpart of nonsingular projective toric varieties. They called this class of manifolds toric manifolds. Since “toric manifold” is used in algebraic geometry for “nonsingular toric variety”, Buchstaber and Panov [3] introduced the term “quasitoric manifold” instead. Quasitoric orbifolds are generalization of quasitoric manifolds and they are studied in [16]. An orbifold with quasitoric boundary is an orbifold with boundary where the boundary is a disjoint union of some quasitoric orbifolds.

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In this article we study the complex cobordism of quasitoric orbifolds. The article is organized as follows. In Section 2, we recall the definition of stable complex structure on an orbifold. In Section 3, we recall the definition of quasitoric orbifold, omniorientation on a quasitoric orbifold and equivariant classification of quasitoric orbifolds. Also we show that a quasitoric orbifold over a simplex is equivariantly homeomorphic to a complex orbifold projective space, see Lemma 3.9. In Section 4, we construct oriented orbifolds with quasitoric boundary. In Section 5, first we show that the orbifolds with quasitoric boundary which are constructed in Section 4 have stable complex structure, see Theorem 5.5. Then we show that a quasitoric orbifold is complex cobordant to finite disjoint copies of complex orbifold projective spaces, see Theorem 5.6. This process produces explicit complex cobordism relations among quasitoric orbifolds. We show that the set of all complex orbifold cobordism classes of complex orbifold projective spaces is not linearly independent, see Observation 5.8. Note that this is in contrast to manifold case. As a particular case when the orbifold singularity is trivial, we get explicit complex cobordism relations among quasitoric manifolds. At the end of Section 5, we give some sufficient conditions to the famous problem of Hirzebruch which asks when a complex cobordism class in the complex cobordism ring Ω^U for manifolds may contain a connected nonsingular algebraic variety. In [21], Andrew Wilfong gives some necessary condition of this problem up to dimension 8. We also compute the Chern numbers of a quasitoric manifold over a simplex, see Example 5.10.

2. Orbifolds

Orbifolds were introduced by Satake [19], who called them V-manifolds. An orbifold is a singular space that is locally look like a quotient of an open subset of Euclidean space by an action of a finite group. The readers are referred to the Section 1.1 in [1] for the definition and basic facts concerning effective orbifolds. Also they may see [12] for an excellent exposition of the foundation of the theory of the reduced differentiable orbifolds.

Similarly as the definition of manifold with boundary, we can talk about orbifold with boundary, see Definition 1.3 in [8]. In this article Druschel studies the orientation on orbifolds in Section 1.

Many concepts in orbifold theory are defined in the context of groupoid, see [1] to enjoy this approach. For example, Section 2.3 in [1] talks about orbifold vector bundle in the language of groupoid. Most relevant example of the orbibundle of an orbifold is its tangent bundle. An explicit description of the tangent bundle of an effective orbifold is given in Section 1.3 of [1].

Definition 2.1. Let Y be a smooth orbifold with the tangent bundle (TY, p_Y, Y) where $p_Y : TY \to Y$ is the projection map.

(1) An almost complex structure on Y is an endomorphism $J : TY \to TY$ such that $J^2 = -Id$.
(2) A stable almost complex (or stable complex) structure on Y is an endomorphism

$$J : TY \oplus (Y \times \mathbb{R}^k) \to TY \oplus (Y \times \mathbb{R}^k) \tag{2.1}$$

such that $J^2 = -Id$ for some positive integer k.

3. Quasitoric orbifolds

In this section we review the definition of quasitoric orbifold following [16]. We also discuss several results on quasitoric orbifolds. An n-dimensional *simple polytope* in $\mathbb{R}^n$ is a convex polytope where exactly n bounding hyperplanes meet at each vertex. A *facet* is a codimension one face of a convex polytope. If P is a convex polytope then we denote the set of all facets of P by $\mathcal{F}(P)$. Let $\mathbb{T}^n = (\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R})/\mathbb{Z}^n$ and $\mathbb{T}_M = (M \otimes_{\mathbb{Z}} \mathbb{R})/M$ for a free $\mathbb{Z}$-module M.