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Rigid frame maps and Booleanization

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ABSTRACT

The article extends the notion of rigidity from algebraic frames to the theory of general frames that are not necessarily algebraic. In particular, the aim of the article is to investigate the relationship between rigid, skeletal, and dense frame homomorphisms, and study the Booleanization of frames in terms of rigidity. Some new concepts on domain preserving, domain reflecting, and strongly skeletal maps are introduced.

1. Background and some basic definitions

Rigid extensions have been studied in the theories of lattice-ordered groups [6], commutative rings [5], and algebraic frames [4] with the main motivation being the study of minimal prime spaces using these extensions. If $L \leq M$ (in any of these settings), then points of the prime space of M map continuously to points of the prime space of L by restriction. This map will not generally send minima to minima, however. Rigidity is meant to remedy this, roughly, by requiring the embedding to reflect orthogonality in a suitable way. For example, for lattice-ordered groups, the embedding is said to be *rigid* if for every $m \in M^+$ there is an $\ell \in L^+$ so that m and ℓ are orthogonal to the same elements. Evidently, this idea does not require that L be a substructure of M, though that is the common application.

In the current article we consider notions related to rigidity for general frames, investigating the interior properties of frames based on these. In particular, we show that rigidity is the precise condition that is required by a dense skeletal map to ensure the ontoness of the corresponding Booleanization. Hence in the category of completely regular frames, these maps capture precisely what is needed for a map to be

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an essential monomorphism. This enhances the result obtained by Banaschewski and Hager [1]. For two elements of a frame, say they are *indistinguishable* if they are orthogonal to the same elements. A frame map is *rigid* if every element of the codomain is indistinguishable from an element of the image.

We do not give all the basic definitions of frame theory, referring the reader to [13] for details. The seminal paper of Isbell [10] in the study of topology via frames continues to be a valuable source of motivations and basic concepts.

Given a frame L and $x \in L$, the *pseudocomplement* of x is defined to be the largest element in L that is disjoint from x, and is denoted by x^*; explicitly in the signature of frames $x^* = \bigvee\{a \in L : a \wedge x = 0\}$. The following are true in any frame:

(1) $(x \vee y)^* = x^* \wedge y^*$.
(2) $(x \wedge y)^* \geq x^* \vee y^*$.
(3) $(x \vee x^*)^* = 0$.

Recall that a *dense element* of a frame is one for which $x^* = 0$, or equivalently $x^{**} = e$. A frame map $h : L \longrightarrow M$ is *dense* if 0 is the only element in L that maps to 0 in M. In the rest of the article we will use various notations for pseudocomplements depending on context, to avoid any confusion:

Given $x \in L$ and $h : L \longrightarrow M$ a frame map,

(1) $x^\perp$ denotes the pseudocomplement in the frame L,
(2) $h(x)'$ the pseudocomplement in the (sub)frame $h(L)$,
(3) $h(x)^*$ the pseudocomplement in M.

Notice that $h(x)' \leq h(x)^*$ and $h(x^\perp) \leq h(x)^*$ always. In general $h(x^\perp) \neq h(x)'$, but dense maps provide a useful special case.

Lemma 1.1. *Let $h : L \longrightarrow M$ be a dense map. For all $x \in L$, $h(x^\perp) = h(x)'$, and hence $h(x^{\perp\perp}) = h(x^\perp)' = h(x)''$.*

Proof. Notice that $h(x^\perp) \wedge h(x) = h(x^\perp \wedge x) = h(0) = 0$. Now, let $l \in L$ with $h(l) \wedge h(x) = 0$, then $h(l \wedge x) = 0$. Since h is dense, $l \wedge x = 0$; so $l \leq x^\perp$. It follows that $h(l) \leq h(x^\perp)$. Therefore, $h(x^\perp) = h(x)'$. □

It will become apparent that pseudocomplementation and double pseudocomplementation play an interesting and crucial role in our theory of rigid maps, and since pseudocomplemented elements are precisely the double pseudocomplemented ones (it is easy to verify that $x^* = x^{***}$ for all elements x of a frame L), we feel it is necessary to highlight these elements by naming them *domain elements*. Historically this term was introduced, in the spatial setting, by Lebesgue and endorsed by Kuratowski and Engelking [7]. Specifically regular opens (closes) are referred to as open (closed) domains. In frame theory, these elements were naturally called "regular" (see Johnstone [11]) however we find that this term can be misleading (and is, indeed, used by others to describe other special elements). In the same vein, we will refer to the collection of all domain elements of a frame L by DL. Clearly, as sets, this is precisely $\mathfrak{B}L = \{x \in L : x = x^{**}\}$ where $\beta_L : L \longrightarrow \mathfrak{B}L$ is the *Booleanization* of L. It is well-known that $\mathfrak{B}L$ is the smallest dense quotient of L, and is indeed a Boolean frame, but it is NOT necessarily a subframe of L (see [11]). Indeed this is only the case if L is extremally disconnected. Moreover $\mathfrak{B}$ is not, in general functorial. We want to think of these elements as a substructure of L, and so DL, the collection of all domain elements will be thought of as a $\wedge$-sub-semilattice of L. Now for subframes, we need to distinguish between the domain elements of the subframe and the domain elements of the larger frame determined by the subframe: Let $N \hookrightarrow M$ be a subframe, then $DN = \{x \in N : x = x''\}$, and $D_M N = \{x \in N : x = x^{**}\}$. The concepts of Booleanization