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Continuous images of linearly ordered continua and compacta

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Dedicated to the memory of Mary Ellen Rudin

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A B S T R A C T

Mary Ellen Rudin has enriched general topology with many remarkable theorems and intricate examples. One of her last contributions was the proof of Nikiel's conjecture that Hausdorff compact monotonically normal spaces are continuous images of linearly ordered compacta. Previously, L. Bruce Treybig and Jacek Nikiel proved that connected locally connected images of linearly ordered compacta are also images of linearly ordered continua. Therefore, images of linearly ordered continua coincide with monotonically normal locally connected continua. Since metric compacta are monotonically normal, this deep result is an extension of the celebrated Hahn–Mazurkiewicz theorem, giving a topological characterization of continuous images of the unit interval of real numbers as locally connected metrizable continua. The purpose of the present paper is to pay tribute to Mary Ellen Rudin by surveying the research in this area realized by a number of topologists during the past hundred years, i.e., since 1914, the year of the publication of the Hahn–Mazurkiewicz theorem.

1. Introduction

Mary Ellen Rudin, during her long and fruitful life, proved many remarkable theorems and constructed surprising examples leaving a lasting imprint on general and set-theoretic topology. Among such theorems is one of her last results (2001), the proof of *Nikiel's conjecture*, that compact monotonically normal spaces coincide with continuous images of linearly ordered compact spaces [60]. Combining this result with impor-

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tant earlier work of L. Bruce Treybig [108] and Jacek Nikiel [58], one obtains a topological characterization of images of linearly ordered continua as locally connected monotonically normal continua. The elegance of this result will please every mathematician, but it does not diminish the importance of Treybig's and Nikiel's work revealing the strange internal structure of these continua. Replacing monotone normality by metrizability, the characterization theorem becomes the classical *Hahn–Mazurkiewicz theorem*, one of the highlights of topology of metric continua, proved independently by Hans Hahn in Vienna [24] and Stefan Mazurkiewicz in Warsaw [54]. The theorem characterizes images of the unit interval $I = [0,1] \subseteq \mathbb{R}$ as *Peano continua*, i.e., metric locally connected continua.

Indeed, since metric spaces are monotonically normal, every Peano continuum X is the image of a surjective mapping $f: C \to X$, for some linearly ordered continuum C. It remains to see that, for X nondegenerate, there is no loss of generality in assuming that $C = I$. Indeed, by a well-known factorization theorem for mappings between compact Hausdorff spaces ([81] or [18], Theorem 6.2.22), f is the composition of a *monotone mapping* $g: C \to Y$ (the fibres $g^{-1}(y)$ of points $y \in Y$ are connected and $g(C) = Y$) and a *light mapping* $h: Y \to X$ (the fibres $h^{-1}(y)$ of points $y \in Y$ are totally disconnected). Since the monotone image of a linearly ordered continuum is again such a continuum, it follows that Y is a nondegenerate linearly ordered continuum. Furthermore, light mappings between nondegenerate continua preserve the weight [40] and thus, $w(Y) = w(X) = \aleph_0$, i.e., Y is a nondegenerate metrizable linearly ordered continuum. The only such continuum is I and thus, $X = h(I)$ is a continuous image of I.

The purpose of the present paper is to pay tribute to Mary Ellen Rudin by a survey of the work on (continuous) images of linearly ordered continua (abbreviated IOC) and linearly ordered compacta (abbreviated IOK) during the past hundred years, i.e., from 1914, the year of the publication of the Hahn–Mazurkiewicz theorem. The paper describes the results of a number of general topologists, who studied IOCs and IOKs, especially after 1960, when one began to realize that beyond the metric case, IOCs and IOKs form surprisingly narrow classes of spaces, having an unexpected structure.

By a *compactum* we mean in this paper a Hausdorff compact space and by a *continuum* we mean a connected compactum. An *ordered compactum* is a compactum that admits a linear ordering such that the topology of the compactum coincides with the induced order topology. *Ordered continua* (also called *generalized arcs*) are connected ordered compacta. They are the nonmetric analogues of $I = [0,1]$. Nondegenerate ordered continua can be characterized as continua having two non-cut points or as locally connected continua irreducible between two points, called the ends. According to other characterizations, ordered continua coincide with chainable locally connected continua [40] and with limits of inverse systems of copies of I with monotone bonding mappings [40,57]. We use the term *arc* only for homeomorphic images of I.

In the literature there are several survey articles devoted entirely or partially to IOCs and IOKs. We mention here [43,109,63] and [53]. There is also a monograph on the subject [73].

2. Peano continua

In the nineteenth century and earlier mathematicians considered the notion of curve to be intuitively clear and not requiring a technical definition. It appears that Camille Jordan in Paris was the first one, who around the year 1880, felt the need for a precise definition. One can find such a definition in his famous textbook *Cours d'Analyse* ([28], page 90). There one reads that a (continuous) curve is the locus of successive positions of a moving point. In the case of a planar motion, it is represented by a system of two equations $x = \phi(t)$ and $y = \psi(t)$, where ϕ and ψ are continuous functions of an independent variable t, which can be thought as representing time, varying between an initial and a terminal value. Using today's terminology, one can say that Jordan defined a *curve* as a metric (Hausdorff) space X that admits a (continuous) mapping of the unit interval I onto X.

That Jordan's definition of a curve is too broad became clear when in 1890 Giuseppe Peano exhibited his celebrated example of a mapping f of I onto the unit square $I \times I \subseteq \mathbb{R}^2$ [76]. This event made topologists to