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Composants of the Stone–Čech remainder of the reals

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ABSTRACT

This is a survey describing Mary Ellen Rudin's theorem about the number of composants of the Stone–Čech remainder of a half-line and subsequent related work, especially the principle of near coherence of filters.

1. Introduction

The purpose of this survey paper is to describe Mary Ellen Rudin's result [21] about the composants of the Stone–Čech remainder of $\mathbb{R}$ and to describe some subsequent developments arising from this result.[1] It is intended for a broad audience, so we devote Section 2 to explaining the notions of composant and Stone–Čech remainder that occur in Rudin's theorem.

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[1] I thank the editors of this special issue, Peter Nyikos and Gary Gruenhage, for inviting me to contribute "a survey for a general audience on Near Coherence of Filters and its applications, especially Mary Ellen's work on composants." I have attempted to comply with the request to write for a general audience.

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Convention 1. All topological spaces in this paper are understood to be Hausdorff spaces. Maps between topological spaces are understood to be continuous unless the contrary is explicitly stated.

2. Background

2.1. Compactifications of the real line

We shall be interested primarily in a particular compactification of the real line $\mathbb{R}$, its Stone–Čech compactification $\beta\mathbb{R}$, but this space is best understood in the context of general compactifications of $\mathbb{R}$ and even of other spaces, so we begin with some generalities.

Definition 2. A *compactification* of a space X is a compact space Y having X as a dense subspace.

We shall not always carefully distinguish a space X from its homeomorphic copies, and so we may apply the term "compactification of X" to any compact space Y with a specified dense embedding of X into Y.

The intuition behind the definition is that Y is obtained from X by adjoining points "at infinity" to serve as limit points for those sequences (or nets or filters) in X that have no limit points in X itself.

The simplest compactification of $\mathbb{R}$, or indeed of any locally compact space, is the one-point compactification, obtained by adjoining a single new point ∞. The open neighborhoods of ∞ are, by definition, the sets $(X - K) \cup \{\infty\}$ for all compact subsets K of X. For example, the one-point compactifications of the line $\mathbb{R}$, the plane $\mathbb{R}^2$, and, more generally, Euclidean spaces $\mathbb{R}^n$ are homeomorphic to the circle, the Riemann sphere, and the n-dimensional spheres, respectively. (Local compactness of X is used to ensure that the one-point compactification is a Hausdorff space.)

Perhaps the most familiar compactification of $\mathbb{R}$ is the two-point compactification, often called the extended real line, obtained by adjoining points $+\infty$ and $-\infty$, one at each "end" of $\mathbb{R}$. A neighborhood base for $+\infty$ (respectively $-\infty$) consists of open half-lines $(a, +\infty]$ (respectively $[-\infty, a)$).

The two-point compactification is more refined than the one-point compactification, in that, instead of lumping together all sequences that "go to infinity" in the sense of escaping from all compact sets, it distinguishes between those that escape to the right and those that escape to the left. Formally, this idea of refinement is reflected in the fact that the identity map of $\mathbb{R}$ extends to a continuous surjection from the two-point compactification to the one-point compactification, mapping both of $\pm\infty$ to the single point ∞, in effect "forgetting" the left-right distinction.

It is possible to refine further. Imagine the real line embedded in the plane as the graph of the sine function. Then among the sequences tending to $+\infty$, some do so at height 1 above the x-axis, some at height -1, some at any intermediate height, and some at oscillating heights. The compactification suggested by this picture would have a copy of the vertical interval $[-1, 1]$ at $+\infty$ (and a similar interval at $-\infty$). (If one makes the situation at infinity more visible by applying the map $(x, \sin x) \mapsto (1/x, \sin x)$, then one has a version of what is often called the Warsaw circle.) A similar construction with the cosine in place of the sine produces a similar compactification. Both of these compactifications refine (in the same sense as above) the two-point compactification, but neither refines the other.

One can continue in this vein, for example, combining the sine and cosine versions by imagining $\mathbb{R}$ embedded in three-dimensional space by $x \mapsto (x, \cos x, \sin x)$ and obtaining a compactification with circles at $\pm\infty$. Sine functions with other periods give more compactifications.

It is reasonable to ask for a finest compactification, one that makes all possible distinctions between ways of going to infinity. More formally, one defines this compactification as follows.

Definition 3. The *Stone–Čech compactification* βX of a space X is a compactification with the property that, if Y is any compactification of X, then the identity map of X extends to a continuous surjection $\beta X \to Y$.