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The basis problem for subspaces of monotonically normal compacta

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ABSTRACT

We prove, assuming Souslin's Hypothesis, that each uncountable subspace of each zero-dimensional monotonically normal compact space contains an uncountable subset of the real line with either the metric, the Sorgenfrey, or the discrete topology.

0. Introduction

In [20], Todorcevic put forward a deep and outstanding program to inspect basis problems in combinatorial set theory. The blueprint can be described as follows: Given a class of structures $\mathcal{S}$,

(1) Identify critical members of $\mathcal{S}$, and
(2) Show that a certain list of critical members is complete.

In other words, one wishes to find a *basis* for $\mathcal{S}$. In a specific category, the only part lacking for such a plan to be precise is to have interpretations for the terms "critical" and "complete".

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In the category of topological spaces, for instance, the morphisms used to interpret completeness may be taken to be embeddings. In this sense, for $\mathcal{S}$ a class in this category, a basis for $\mathcal{S}$ is a subclass $\mathcal{S}_0$ such that each member of $\mathcal{S}$ contains a homeomorphic copy of a member of $\mathcal{S}_0$. Ensuring that the choice of $\mathcal{S}_0$ is minimal (in the sense of cardinality), the members of $\mathcal{S}_0$ are clearly then to be interpreted as the critical objects of $\mathcal{S}$.

Let $\mathcal{S}$ now be the class of uncountable first countable regular spaces. A basis conjecture for $\mathcal{S}$ was first formulated by Gruenhage in [6]. This has come to be called the *3-element basis conjecture for uncountable first countable regular spaces*, as it states that it is consistent that each such space contains a set of reals of cardinality $\aleph_1$ with either the metric, the Sorgenfrey, or the discrete topology. If true, this conjecture would afford a deep understanding of the structure of regular spaces; in fact, the truth of even partial instances of it would have far-reaching consequences and settle long-open problems in set-theoretic topology.

To state the conjecture, let us denote by

- $D(\omega_1)$, the discrete space on ω_1,
- B, a fixed $\aleph_1$-dense subset of the open unit interval, and
- $B \times \{0\}$, the corresponding subspace of the split interval (or, double arrow space).

The 3-element Basis Conjecture. *PFA implies that the class of uncountable first countable regular spaces has a three element basis consisting of $D(\omega_1)$, B, and $B \times \{0\}$.*

An excellent discussion of the basis conjecture, specifically for perfectly normal compacta, can be found in [8].

T. Eisworth proved in [3] that each separable Hausdorff space which is a continuous image of an ordered compactum admits a continuous and at-most 2-to-1 map onto a metric space. Gleaning out ideas from [5] and [6], it follows from this result that PFA implies that the class of uncountable subspaces of monotonically normal compacta has a 3-element basis. In this paper, we prove that actually Souslin's Hypothesis is sufficient to guarantee that each uncountable space having a zero-dimensional monotonically normal compactification contains either an uncountable discrete subspace, an uncountable subspace of the real line, or an uncountable subspace of the Sorgenfrey line. In Section 2 we employ towards this end a direct combinatorial analysis, and in Section 3 we use continuum theoretic techniques to obtain finer structure results for monotonically normal compacta. Finally, we conclude the paper by indicating directions for future research.

This work could be seen as a simple exhibit of the power of M.E. Rudin's latest breakthrough: a proof of Nikiel's conjecture.

1. Preliminaries

The reader is referred to [4] for basic facts stated without proof. Unless stated otherwise, all sets are assumed to be uncountable, and all spaces Hausdorff. Maps and images refer to continuous functions and their images, respectively. A compactum is a compact space, and a continuum is a connected compactum. A linearly ordered set equipped with the usual open interval topology is called an *ordered space*.

Our departing point for the study of monotonically normal compacta is Nikiel's conjecture [14], proved by M.E. Rudin [18]. It states that the class of monotonically normal compacta coincides with the class of continuous images of ordered compacta. We therefore use the terms "compact monotonically normal" and "image of a compact ordered space" interchangeably.

Prototypical ordered compacta include the Cantor set in the metric category and the split interval in the separable non-metric category. The split interval is a classical space first described by Alexandroff, and otherwise known by the more common name, the double arrow space. Recall that each compact metric space is an image of the Cantor set. Below we give short proofs of folklore results which establish an analogue