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Between countably compact and ω-bounded [☆]

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A B S T R A C T

Given a property P of subspaces of a T_1 space X, we say that X is P-*bounded* iff every subspace of X with property P has compact closure in X. Here we study P-bounded spaces for the properties $P \in \{\omega D, \omega N, C_2\}$ where $\omega D \equiv$ "countable discrete", $\omega N \equiv$ "countable nowhere dense", and $C_2 \equiv$ "second countable". Clearly, for each of these P-bounded is between countably compact and ω-bounded.
We give examples in ZFC that separate all these boundedness properties and their appropriate combinations. Consistent separating examples with better properties (such as: smaller cardinality or weight, local compactness, first countability) are also produced.
We have interesting results concerning ωD-bounded spaces which show that ωD-boundedness is much stronger than countable compactness:

- Regular ωD-bounded spaces of Lindelöf degree $< \mathrm{cov}(\mathcal{M})$ are ω-bounded.
- Regular ωD-bounded spaces of countable tightness are ωN-bounded, and if $\mathfrak{b} > \omega_1$ then even ω-bounded.
- If a product of Hausdorff spaces is ωD-bounded then all but one of its factors must be ω-bounded.
- Any product of at most $\mathfrak{t}$ many ωD-bounded spaces is countably compact.

As a byproduct we obtain that regular, countably tight, and countably compact spaces are discretely generated.

1. Introduction

The work we report on in this paper can be considered as complementary to that in [10] where certain natural boundedness properties that are strengthenings of the ω-bounded property had been investigated. Here we go in the opposite direction in the sense that we study boundedness properties that are weaker

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than ω-bounded but still stronger than countably compact. All spaces we consider in this work are assumed to be T_1. Hence for them the two usual definitions of countable compactness (the covering definition and the accumulation point definition) coincide.

Definition 1.1. Given a property P of subspaces of a space X, we say that X is *P-bounded* iff every subspace of X with property P has compact closure in X.

In this paper our aim is to study P-bounded spaces for the following choices of a property P of subspaces:

- $P \equiv$ "countable discrete" (in short: ωD)
- $P \equiv$ "countable nowhere dense" (in short: ωN)
- $P \equiv$ "second countable" (in short: C_2)

Let us note that, as second countable spaces are separable, we clearly have

$$C_2\text{-bounded} \equiv \omega C_2\text{-bounded} \equiv \omega C_1\text{-bounded}.$$

Here, of course, ωC_2 stands for "countable and second countable" while C_1 abbreviates "first countable".

The referee has pointed out to us that the property of *total countable compactness* nicely fits with these properties. We recall that a space X is *totally countably compact* iff every infinite subset $A \subset X$ has an infinite subset $B \subset A$ that has compact closure in X. (See J. Vaughan's article [15] in the Handbook of Set-Theoretic Topology.)

Since any infinite set in a Haudorff space has an infinite discrete subset, ωD-bounded Haudorff spaces are totally countably compact. We shall show below, however, that this implication fails for T_1 spaces.

Discrete subspaces of a *crowded* space are clearly nowhere dense, hence it is trivial that both ωN-bounded crowded spaces and C_2-bounded spaces are ωD-bounded. The diagram of Fig. 1 gives a visual summary of the easy implications between (total) countable compactness, ω-boundedness and our new boundedness properties and their natural combinations.

It should be mentioned that the diagram on page 587 of Vaughan's article [15] mentions nothing between ω-bounded and totally countably compact, thus the new classes of spaces treated in this paper neatly extend his diagram.

2. Separation in ZFC

The aim of this section is to show that all the classes of spaces appearing in diagram of Fig. 1 are distinct. In other words, we claim that no new arrows can be added to that diagram.

In fact, we are going to produce, in ZFC, zero-dimensional crowded spaces that separate any two of these classes. Before we turn to constructing these examples, however, we need to do some preliminary work.

Definition 2.1. Given a property P of subspaces of a space X and a fixed subset $Y \subset X$, we write

$$[Y]^P = \{A \subset Y : A \text{ has property } P\},$$

and

$$\mathrm{cl}_X^P(Y) = \bigcup\{\overline{A} : A \in [Y]^P\}.$$

Also, we say that Y is *P-closed* in X if $Y = \mathrm{cl}_X^P(Y)$. (As usual, we shall omit the subscript X when it is clear from the context.)