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A special point from ♢ and strongly ω-bounded spaces

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A Tychonoff space is CNP if it is a P-set in its Stone–Čech compactification. We are interested in the question of whether the property of being a CNP space is finitely productive. A space X is strongly ω-bounded if every σ-compact subset of X has compact closure in X. In this paper, we show that the existence of two CNP spaces whose product is not CNP is equivalent to the existence of a space which is not strongly ω-bounded but which is the union of two subsets each of which is strongly ω-bounded. We use ♢ to construct a special point in $\beta(\omega\times(\beta\omega\setminus\omega))$ and use that point to find a non-strongly ω-bounded space which is a union of two strongly ω-bounded subsets.

1. Introduction

In this paper all given spaces are assumed to be Tychonoff. If X is a space, βX denotes the Stone–Čech compactification of X, and X^* denotes the Stone–Čech remainder $\beta X \setminus X$ of X. If K is a compact space and $f: X \to K$ is a continuous function, then the Stone extension of f is denoted f^β. If $f: X \to \mathbb{R}$ is a function, we denote by $Z(f)$ the zero-set $\{x \in X : f(x) = 0\}$ of f.

Recall that a space X is ω-*bounded* if every countable subset of X has compact closure in X. A strengthening of the notion of ω-boundedness is the notion of strong ω-boundedness. A space X is *strongly* ω-*bounded* if every σ-compact subset of X has compact closure in X. Clearly every compact space is strongly ω-bounded, every strongly ω-bounded space is ω-bounded, and every ω-bounded space is countably compact. The following examples distinguish these properties.

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Examples 1.1.

(1) The ordinal space ω_1 is strongly ω-bounded but not compact.
(2) The Σ-product in 2^{ω_1} with base point $\langle 0\rangle$ is ω-bounded, but, since the σ-product with base point $\langle 0\rangle$ is a dense σ-compact subset, the Σ-product is not strongly ω-bounded.
(3) If $p \in \omega^*$, then $\beta\omega \setminus \{p\}$ is countably compact but not ω-bounded.

A space X has the *countable neighborhood property*, or is a *CNP* space, if whenever $\{U_n : n \in \omega\}$ is a countable collection of neighborhoods of X in its Stone–Čech compactification βX, the intersection $\cap_{n\in\omega} U_n$ is a neighborhood of X in βX. (See [1].) In other words, X is a CNP space if it is a P-set in its Stone–Čech compactification. From these definitions, the following is clear.

Observation 1.2. *The space X is CNP if and only if X^* is strongly ω-bounded.*

If BX is a compactification of the space X and $\iota: X \to BX$ is the injection map, then it is well-known that $\iota^\beta(X^*) = BX \setminus X$ and the restriction $\iota^\beta \restriction X^*$ is a perfect map. Since perfect maps preserve compactness and, therefore, σ-compactness under both images and inverse images, it follows that X^* is strongly ω-bounded if and only if $BX \setminus X$ is strongly ω-bounded. Therefore, we get the following.

Proposition 1.3. *([1]) Suppose that X is a space. Then the following statements are equivalent.*

(1) *X is a CNP space.*
(2) *X is a P-set in some compactification of X.*
(3) *X is a P-set in every compactification of X.*
(4) *X^* is strongly ω-bounded.*
(5) *There is a compactification BX of X such that $BX \setminus X$ is strongly ω-bounded.*
(6) *For every compactification BX of X, $BX \setminus X$ is strongly ω-bounded.*

Our interest in strong ω-boundedness and CNP spaces arose from the following question, which remains open.

Question 1.4. *(Barr, Kennison, Raphael [1]) Suppose that X and Y are CNP Lindelöf spaces. Is the product $X \times Y$ either CNP or Lindelöf?*

The outline of the paper is as follows. In Section 2, we describe examples of CNP spaces and give some properties of Lindelöf CNP spaces. In Section 3 we construct, assuming $\diamondsuit$, a point p in the Stone–Čech remainder Y^* of the space $\omega \times \omega^*$ with the property that p is a remote point of Y and a P-point of Y^*. Section 4 deals with the contrasts between ω-boundedness and strong ω-boundedness. In particular, assuming $\diamondsuit$ a space is given which is not strongly ω-bounded but which is the union of two strongly ω-bounded subsets. In Section 5 we discuss products of CNP spaces and show that, assuming $\diamondsuit$, the property of being CNP is not finitely productive.

For background material, see [3] and [4].

2. Lindelöf CNP spaces

In this section we give some examples of CNP Lindelöf spaces. We start with a characterization of such spaces.