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Forcing with a coherent Souslin tree and locally countable subspaces of countably tight compact spaces

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ABSTRACT

We apply a method of Todorcevic in showing that $PFA(S)$ implies that the coherent Souslin tree S forces various strengthenings of the assertion that every locally countable space of size $\aleph_1$ that has a countably tight compactification must be σ-discrete.

1. Introduction

The third author in 2001 developed technology for analyzing the forcing by a fixed Souslin tree S over a model obtained by iterating proper posets that preserve S. Parts of this work were circulated in the form of notes such as [13] and have been presented at several conferences.[3] Now this work is finally available

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[3] Such as, for example, the three lectures at the Erice conference on set-theoretic topology in 2008 (see [15]).

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in printed form in [14]. We shall use this technology to deduce a particular topological principle about subspaces of compact countably tight spaces.

The proof that *PFA*$(S)[S]$ *implies* $\sum$ that we give here uses the same machinery and notation as the proof in [11] that *PFA*$(S)[S]$ *implies locally compact normal spaces are* $\aleph_1$*-collectionwise Hausdorff.* That machinery is not so elegant as that in [14], but it has the advantage of already being published, so that the reader who wants more details can find them there. We just make one small improvement over [11]: improving coding so as to obtain "σ-discrete", rather than "σ-discrete on a club". The other differences are related to the fact that we face two entirely different topological situations. It is quite remarkable that the set-theoretic techniques are the same for both.

The [11] machinery (based on [13]) is easier to understand for topologists not so expert at forcing with elementary submodels as side conditions. We will take advantage of [11] actually being published to omit some explanatory material and details available there, since our proof here closely resembles the proof there. (In fact, the proof there was inspired by the idea of the proof here.)

We define *coherent* Souslin trees below. At this point, all the reader need know is that they may be obtained from $\diamondsuit$. Given a coherent Souslin tree S, PFA(S) is the restriction of PFA to those proper posets preserving (the Souslinity of) S. Since countable support iterations of such posets preserve S [9], PFA(S) may be obtained by the usual Laver-diamond [3] method from a supercompact cardinal. We use the notation PFA$(S)[S]$ *implies* Φ to mean *given any model of* PFA(S)*, forcing over it with* S *yields a model of* Φ. We say Φ *holds in a model of form* PFA$(S)[S]$ if forcing with S over a particular model of PFA(S) yields a model of Φ.

Definition 1. An S**-space** is a hereditarily separable, T_3, non-Lindelöf space.

The double use of 'S' is unfortunate, but it will always be clear which S is intended. MA_{ω_1} implies there are no compact S-spaces [10]; PFA implies there are no S-spaces [16].

Setting out the context for what we will be doing, we recall that Todorcevic in 2001 proved the following result (see [14]).

Proposition 1. *([14])* PFA$(S)[S]$ *implies there are no compact* S*-spaces.*

One of the crucial parts of the proof is contained in the following result.

Proposition 2. *([14])* PFA$(S)[S]$ *implies compact countably tight spaces are sequential.*

A particular application of this result was given in [5]:

Proposition 3. *([5]) There is a model of form* PFA$(S)[S]$ *in which every locally compact perfectly normal space is paracompact.*

The use of Proposition 2 in the proof of Proposition 3 was the application of an equivalent version of the result stated in the Abstract:

Proposition 4. PFA$(S)[S]$ *implies that in a compact countably tight space, locally countable subspaces of size* $\aleph_1$ *are* σ*-discrete.*

This follows from Proposition 2 plus:

Proposition 5. PFA$(S)[S]$ *implies that in a compact sequential space, locally countable subspaces of size* $\aleph_1$ *are* σ*-discrete.*