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Convex sets in modules over semifields

Karl-Ernst Biebler

Ernst-Moritz-Arndt-University, 17489, Greifswald, Germany

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ABSTRACT

Semifields are commutative preordered rings for which a weak invertibility of the multiplication is defined. Examples are the Stone algebras and the extended complete vector lattices. Convex sets, strong convex sets and absolute convex sets are defined and studied in semifield modules. This is applied to investigations concerning the problem of the semifield-normability of semifield-lineartopological semifield-modules.

1. Complete vector lattices and semifields

Let E be a complete vector lattice with weak order unit 1. It contains the complete Boolean algebra I_E of its idempotent elements. Following the Stonean representation theorem, every Boolean algebra $\mathbb{B}$ is isomorphic to the system ordered by inclusion of all the open–closed subsets of a zero-dimensional topological Hausdorff space $Q(\mathbb{B})$. This representation space is extremally disconnected iff $\mathbb{B}$ is a complete Boolean algebra. $\mathfrak{C}_\infty(Q(I_E))$ denotes the set of all continuous functions defined on $Q(I_E)$ with values in $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. In addition, it is required of each function of $\mathfrak{C}_\infty(Q(I_E))$, that it has the values $+\infty$ and $-\infty$ solely on a (dependent on the function) nowhere dense subset of the domain $Q(I_E)$. In $\mathfrak{C}_\infty(Q(I_E))$, the algebraic operations are defined as follows: Let $\bar{a}_1$ and $\bar{a}_2$ elements of $\mathfrak{C}_\infty(Q(I_E))$. Then there exists an open extremally disconnected everywhere dense subset $G \subset Q(I_E)$, on which $\bar{a}_1(t)$ as well as $\bar{a}_2(t)$ is finite. The sum $b'(t) := \bar{a}_1(t) + \bar{a}_2(t)$ of $\bar{a}_1$ and $\bar{a}_2$ is defined on G. Using an extension theorem of Taimanov [14], this is possible because $Q(I_E)$ is extremally disconnected, b' can be extended uniquely and continuously to a function $\bar{b} \in \mathfrak{C}_\infty(Q(I_E))$. This way $\bar{a}_1 + \bar{a}_2 := \bar{b}$ is defined. One obtains the product of $\bar{a}_1$ and $\bar{a}_2$ analogously. The real numbers are identified with the constant functions of $\mathfrak{C}_\infty(Q(I_E))$. Scalar multiplication is then a

E-mail address: kebiebler@web.de.

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special case of the ring multiplication. A preorder is established in $\mathfrak{C}_\infty(Q(I_E))$ by $\mathfrak{C}_\infty(Q(I_E)) \ni \bar{a} \geq 0$ iff $\bar{a}(t) \geq 0$ for all $t \in Q(I_E)$. $\mathfrak{C}_\infty(Q(I_E))$ is a complete vector lattice.

Definition 1.1 *(Fundament).* A subset M of a complete vector lattice E is called a fundament, when the following applies:

(1) M is a linear subspace of E.
(2) $\forall a \in E \; \forall b \in M \; (|a| \leq |b| \;\Rightarrow\; a \in M)$.
(3) In E exists no element that is different from zero and disjoint to M.

Definition 1.2 *(Trace).* Let E a complete vector lattice.
Then $e_a := \sup\{1 \wedge n\,|a| \;:\; n \in \mathbb{N}\}$ is called the trace of $a \in E$.

Considering E as a vector lattice of functions, e_a would then just be the support of the function a. However, we keep the term used by Wulich [16] because it is also used by the reference literature.

Theorem 1.1 *(Representation theorem for complete vector lattices). Every complete Vector lattice E is algebraic and vector lattice-isomorphic to a fundament of $\mathfrak{C}_\infty(Q(I_E))$. The isomorphism can be chosen so that the weak order unit $1 \in E$ is transferred in the function $1(t) \equiv 1$ from $\mathfrak{C}_\infty(Q(I_E))$. The image of E already contains all the continuous bounded functions of $\mathfrak{C}_\infty(Q(I_E))$, which will be labelled $\mathfrak{C}(Q(I_E))$.*

Every complete vector lattice E can be embedded isomorphically into $\mathfrak{C}_\infty(Q(I_E))$. The function space $\mathfrak{C}_\infty(Q(I_E))$ is already uniquely characterized by I_E. Therefore different complete vector lattices with the same underlying Boolean algebra of idempotent elements have the same representation space $\mathfrak{C}_\infty(Q(I))$.

Example 1. $\mathfrak{C}_\infty(\beta\mathbb{N})$ and the space $\mathfrak{C}(\beta\mathbb{N})$ of the bounded continuous real functions defined on $\beta\mathbb{N}$ possess the power set of $\mathbb{N}$ as a common underlying Boolean algebra. The extension theorem of Taimanov [14] gives the justification that the complete vector lattice $\mathfrak{s}$ of all real sequences and the complete vector lattice $\mathfrak{m}$ of all bounded real sequences can be identified with $\mathfrak{C}_\infty(\beta\mathbb{N})$ or $\mathfrak{C}(\beta\mathbb{N})$, respectively.

Definition 1.3 *(Extended complete vector lattice).* A complete vector lattice E is called an extended complete vector lattice if its image under the embedding in $\mathfrak{C}_\infty(Q(I_E))$ due to Theorem 1.1 coincides with this vector lattice of functions.

Definition 1.4 *(Partially ordered commutative associative ring).* ([15]) A vector lattice M is called a partially ordered commutative ring, when the following conditions are fulfilled by M:

(1) For every pair $(a, b) \in M \times M$ a product $a \cdot b \in M$ is defined.
(2) This product owns the characteristics associativity, commutativity and distributivity.
(3) A unit element of the multiplication exists.
(4) From $a \geq 0$ and $b \geq 0$ results $a \cdot b \geq 0$.
(5) For any given $\lambda \in \mathbb{R}$ the relationship $(\lambda x) \cdot y = \lambda(x \cdot y)$ is valid.

As already discussed, in any extended complete vector lattice E a multiplication can be defined. With the mapping $\mathbb{R} \ni \lambda \to 1 \cdot \lambda \in E$, the real numbers will be identified with the constant functions of the representation space $\mathfrak{C}_\infty(Q(I_E))$. Thus, one obtains the interconnection to the scalar multiplication in E. The unit of the multiplication and the weak order unit 1 coincide with the function $1(t) \equiv 1$.

Theorem 1.2. *Every extended complete vector lattice is a commutative partially ordered ring.*