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A group topology on the real line that makes its square countably compact but not its cube

Ana Carolina Boero [a,*,1], Irene Castro Pereira [b], Artur Hideyuki Tomita [c,2]

[a] *Federal University of ABC (UFABC), Rua Abolição, S/N, CEP 09210-180, Santo André – SP, Brazil*
[b] *Instituto de Ciências Exatas e Naturais, Universidade Federal do Pará, Rua Augusto Corrêa, 1, CEP 66075-110, Belém – PA, Brazil*
[c] *Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão, 1010, CEP 05508-090, São Paulo – SP, Brazil*

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ABSTRACT

Under $\mathfrak{p} = \mathfrak{c}$, we show that it is possible to endow the additive group of the real line with a Hausdorff group topology that makes its square countably compact but not its cube.

1. Introduction

1.1. Some history

Comfort and Ross [6] showed that the product of pseudocompact groups is pseudocompact. This is not true in general for pseudocompact spaces. There is even a countably compact space whose square is not pseudocompact (see [8]). This motivated the question whether the same would be true for countably compact groups.

* Corresponding author.
E-mail address: ana.boero@ufabc.edu.br (A.C. Boero).

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The first solution to this problem was obtained by van Douwen [7]. He showed under Martin's Axiom the existence of two countably compact groups whose product is not countably compact. Using a different technique, Hart and van Mill [10] constructed under Martin's Axiom restricted to countable posets a countably compact group whose square is not countably compact. Ginsburg and Saks [9] showed that if the $2^{\mathfrak{c}}$th power of a topological space X is countably compact, then every power of X is countably compact.

Comfort [5] asked if there exists, for every (not necessarily infinite) cardinal number $\alpha \geq 2^{\mathfrak{c}}$, a topological group G such that G^γ is countably compact for all cardinals $\gamma < \alpha$, but G^α is not countably compact. Tomita [11] showed it is consistent with ZFC that each cardinal not greater than $2^{\mathfrak{c}}$ answers Comfort's question affirmatively.

Boero and Tomita [2] showed under $\mathfrak{p} = \mathfrak{c}$ that it is possible to endow the free Abelian group of size continuum with a group topology that makes its square countably compact. In this article, we show under $\mathfrak{p} = \mathfrak{c}$ that it is possible to endow the additive group of the real line with a Hausdorff group topology that makes its square countably compact but not its cube.

1.2. Basic results, notation and terminology

In what follows, all group topologies are assumed to be Hausdorff. We recall that a topological space X is *countably compact* if every infinite subset of X has an accumulation point.

The following definition was introduced in [1] and is closely related to countable compactness.

Definition 1.1. Let p be a free ultrafilter on ω and let $\{x_n : n \in \omega\}$ be a sequence in a topological space X. We say that $x \in X$ is a *p-limit point* of $\{x_n : n \in \omega\}$ if, for every neighborhood U of x, $\{n \in \omega : x_n \in U\} \in p$. In this case, we write $x = p - \lim\{x_n : n \in \omega\}$.

The set of all free ultrafilters on ω will be denoted by ω^*. It is not difficult to show that a topological space X is countably compact if, and only if, each sequence in X has a p-limit point, for some $p \in \omega^*$.

Proposition 1.2. *If $p \in \omega^*$ and $\{X_i : i \in I\}$ is a family of topological spaces, then $(y_i)_{i\in I} \in \prod_{i\in I} X_i$ is a p-limit point of $\{(x_i^n)_{i\in I} : n \in \omega\} \subset \prod_{i\in I} X_i$ if, and only if, $y_i = p - \lim\{x_i^n : n \in \omega\}$ for every $i \in I$.*

Proposition 1.3. *Let G be a topological group and $p \in \omega^*$.*

(1) *If $\{x_n : n \in \omega\}$ and $\{y_n : n \in \omega\}$ are sequences in G and $x, y \in G$ are such that $x = p - \lim\{x_n : n \in \omega\}$ and $y = p - \lim\{y_n : n \in \omega\}$, then $x + y = p - \lim\{x_n + y_n : n \in \omega\}$;*
(2) *If $\{x_n : n \in \omega\}$ is a sequence in G and $x \in G$ is such that $x = p - \lim\{x_n : n \in \omega\}$, then $-x = p - \lim\{-x_n : n \in \omega\}$.*

If A is a set, then $[A]^\omega = \{X \subset A : |X| = \omega\}$ and $[A]^{<\omega} = \{X \subset A : |X| < \omega\}$.

A *pseudointersection* of a family $\mathcal{G}$ of sets is an infinite set that is almost contained in every member of $\mathcal{G}$. We say that a family $\mathcal{G}$ of infinite sets has the *strong finite intersection property* (SFIP, for short) if every finite subfamily of $\mathcal{G}$ has infinite intersection. The *pseudointersection number* $\mathfrak{p}$ is the smallest cardinality of any $\mathcal{G} \in [\omega]^\omega$ with SFIP but with no pseudointersection.

We denote the set of positive natural numbers by $\mathbb{N}$, the integers by $\mathbb{Z}$, the rationals by $\mathbb{Q}$ and the reals by $\mathbb{R}$. The unit circle group $\mathbb{T}$ will be identified with the metric group $(\mathbb{R}/\mathbb{Z}, \delta)$ where δ is given by $\delta(x + \mathbb{Z}, y + \mathbb{Z}) = \min\{|x - y + a| : a \in \mathbb{Z}\}$ for every $x, y \in \mathbb{R}$. Given a subset A of $\mathbb{T}$, we will denote by $\delta(A)$ the diameter of A with respect to the metric δ.

The set of all non-empty open arcs of $\mathbb{T}$ will be denoted by $\mathcal{B}$. If $A \in \mathcal{B} \setminus \{\mathbb{T}\}$ and $n \in \mathbb{N}$, we will denote by A/n the open arc of $\mathbb{T}$ centered at the middle point of A with length $\delta(A)/n$.