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Selective games on binary relations

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ABSTRACT

We present a unified approach, based on dominating families in binary relations, for the study of topological properties defined in terms of selection principles and the games associated to them. Among other applications, this general framework allows us to obtain a number of results on the preservation of such properties in products provided that the second player has a winning strategy in the associated game played on one of the coordinate spaces.

0. Introduction

Classical games introduced by Berner and Juhász [10], Galvin [17], Gruenhage [20] and Telgársky [34,35] associated with diverse topological properties, as well as several subsequent games, can be considered in a single unifying framework based on the notion of a *relation*, defined below in Definition 2.1, and of a *dominating family* for a relation, defined below in Definition 2.2. We define these games in Definition 3.3. This framework subsumes and clarifies several isolated theorems about topological games.[2]

One of the two main phenomena about these topological games that is addressed here is: Consider a topological property, E. For many examples of E one can find spaces X and Y each of which has the property E, while the product space $X \times Y$ does not have the property E. Define a space X to be

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2 Which specific theorems in the literature are affected will be presented later in the paper.

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productively E (or *productive with respect to* E) if, for each space Y that has property E, also $X \times Y$ has the property E. For some properties E it is a significant mathematical problem to characterize the spaces X that are productively E. For several isolated examples of topological games G it has been found that, if a certain player of the game G on a space X has a winning strategy, then X is productively E.

We study the productivity of properties E in this abstract context. The four properties E we consider — which will be described in detail after Definition 3.9 — are as follows:

E_1: TWO has a winning strategy in the game G;
E_2: ONE does not have a winning strategy in the game G;
E_3: A selective version of a certain countability hypothesis holds;
E_4: A certain countability hypothesis holds.

For the properties we consider it will be the case that

$$\mathsf{E}_1 \Rightarrow \mathsf{E}_2 \Rightarrow \mathsf{E}_3 \Rightarrow \mathsf{E}_4,$$

where sometimes, but not always, an implication is reversible.

The nature of our theorems is as follows:

- A relation in the class E_1 is productively E_1.
- A relation in the class E_1 is productively E_3.
- A relation in the class E_1 is productively E_4.

It is curious that as of yet the techniques for the other implications do not seem to produce the implication that a relation in the class E_1 is productively E_2. Some questions related to this issue will be posed at the end of the paper.

These results about products bring us to analyzing situations in which TWO has a winning strategy in a game, and the second of the two main phenomena addressed here: In many instances of games G it is known that, if player TWO has a winning strategy in the game G, then player TWO has a winning strategy in a game G', in which the winning condition for TWO appears more stringent.

This paper is organized as follows: After establishing notational conventions in Section 1 we introduce a general framework for the theory regarding E_1 in Section 2. In Section 3 we translate classical duality results on games to the new framework. In Sections 4 and 5 we prove product theorems. Sections 6 and 7 are dedicated to situations in which the existence of a winning strategy for player TWO in a certain game turns out to be equivalent to the same condition in other games that are seemingly more difficult for TWO. In Section 8 we study conditions under which a Lindelöf-like property is equivalent to ONE not having a winning strategy in the selective game associated to the property being considered. Section 9 contains some final remarks about the results presented in the paper.

1. Notational conventions

Throughout our paper X denotes the underlying set of a topological space and τ denotes the ambient topology on the space. Whenever a second topological space Y is involved, its topology will be denoted by ρ. Unless explicitly stated otherwise, we do not make any assumptions about separation hypotheses on the topological spaces in our results.

For a space X and a point $x \in X$, we write $\tau_x = \{U \in \tau : x \in U\}$. For a subset A of X, the closure of A in X is denoted by $\overline{A}$. The set of all compact subsets of X is denoted by $K(X)$.