Topology and its Applications 192 (2015) 98–112

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

When certain prime ideals in rings of continuous functions are minimal or maximal

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ARTICLE INFO

Article history:
Received 9 February 2014
Accepted 19 May 2014
Available online 29 May 2015

MSC:
06D22
13A15
54G10
54G15

Keywords:
P-frame
Quasi P-frame
Almost P-frame
Quasi cozero complemented frame
Quasi m-space

ABSTRACT

A Tychonoff space X is called a quasi m-space if every prime d-ideal of the ring $C(X)$ is either a maximal ideal or a minimal prime ideal. These spaces were characterised by Azarpanah and Karavan. In this paper we look at some properties of these spaces from a ring-theoretic perspective. We observe, for instance, that among subspaces which inherit this property are (i) cozero subspaces, (ii) dense z-embedded subspaces, and (iii) regular-closed subspaces among the normal quasi m-spaces. The ring-theoretic approach actually yields the above results within the broader context of frames. The latter part of the paper discusses completely regular frames L for which every prime z-ideal in the ring $\mathcal{R}L$ is a maximal ideal or a minimal prime ideal.

1. Introduction

Throughout the paper all rings considered are commutative with identity 1. We denote the annihilator of a set S by $\mathrm{Ann}(S)$, and abbreviate $\mathrm{Ann}(\{a\})$ as $\mathrm{Ann}(a)$. Double annihilators are written as $\mathrm{Ann}^2(S)$. An ideal I of a ring A is called a *d-ideal* if $\mathrm{Ann}^2(a) \subseteq I$, for every $a \in I$. On the other hand, I is called a *z-ideal* if whenever $a \in I$ and b is an element of A contained in every maximal ideal containing a, then $b \in I$. The symbols $\mathrm{Spec}(A)$, $\mathrm{Max}(A)$ and $\mathrm{Min}(A)$ have their usual meaning; namely, the set of prime, maximal and minimal prime ideals of A, respectively. We write $\mathrm{Spec}_d(A)$ and $\mathrm{Spec}_z(A)$ for the set of prime d-ideals and prime z-ideals of A, respectively.

Consider the following conditions on a ring A:

(dMin) $\quad \mathrm{Spec}_d(A) \subseteq \mathrm{Min}(A)$

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http://dx.doi.org/10.1016/j.topol.2015.05.073
0166-8641/

(dMax)	$\mathrm{Spec}_d(A) \subseteq \mathrm{Max}(A)$
(dMM)	$\mathrm{Spec}_d(A) \subseteq \mathrm{Min}(A) \cup \mathrm{Max}(A)$
(zMin)	$\mathrm{Spec}_z(A) \subseteq \mathrm{Min}(A)$
(zMax)	$\mathrm{Spec}_z(A) \subseteq \mathrm{Max}(A)$
(zMM)	$\mathrm{Spec}_z(A) \subseteq \mathrm{Min}(A) \cup \mathrm{Max}(A)$.

One of our goals is to characterise frames L such that, for each of these conditions, the ring $\mathcal{R}L$ satisfies that condition. In the class of reduced rings (i.e. rings with no nonzero nilpotent elements), three of these are all equivalent, and are equivalent to the property of being von Neumann regular (vNR). Precisely,

$$\text{zMin} \iff \text{zMax} \iff \text{dMax} \iff \text{vNR}.$$

Indeed, (zMin) is equivalent to (vNR) because maximal ideals are z-ideals, and a reduced ring is von Neumann regular if and only if every maximal ideal is minimal prime. Next, (zMax) implies (dMax) because every d-ideal is a z-ideal; (dMax) implies (vNR) because every minimal prime ideal is a d-ideal; and, finally, (vNR) implies (zMax) because every prime ideal in a von Neumann regular ring is a maximal ideal. Thus, $\mathcal{R}L$ satisfies any (and hence all) of these three if and only if L is a P-frame because L is a P-frame if and only if $\mathcal{R}L$ is von Neumann regular [7].

We shall see that $\mathcal{R}L$ satisfies (dMin) precisely if L is cozero complemented (Proposition 3.1). The more substantive results concern those L for which $\mathcal{R}L$ satisfies (dMM) and those for which $\mathcal{R}L$ satisfies (zMM). The latter kind we will call quasi P-frames, and the definition we use is motivated by the definition of quasi P-spaces [22]. Although P-frames generalise P-spaces in the sense that a Tychonoff space is a P-space precisely when the frame of its open sets is a P-frame, it has recently been shown by Ball, Walters-Wayland and Zenk [4] that, in stark contrast with P-spaces, there are P-frames with quotients which are not P-frames. In the last part of this article we examine how far the theory of quasi P-frames parallels that of quasi P-spaces as defined by Henriksen, Martínez and Woods [22].

2. Preliminaries

2.1. Frames

Our reference for the general theory of frames is [29]. Throughout, L stands for a completely regular frame. We view the Stone–Čech compactification of L as the frame of completely regular ideals of L. We denote by $j_L\colon \beta L \to L$ the join map $J \mapsto \bigvee J$. The right adjoint of j_L is here denoted by r_L. Recall that, for any $a \in L$, $r_L(a) = \{x \in L \mid x \prec\!\prec a\}$. We write, as usual, $\mathrm{Coz}\, L$ for the set of cozero elements of L. If $a, b \in \mathrm{Coz}\, L$, then $r_L(a \vee b) = r_L(a) \vee r_L(b)$.

By a *point* of a frame we mean a prime element, that is, an element $p < 1$ such that for any a and b in the frame, $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. We denote by $\mathrm{Pt}(L)$ the set of all points of L. We remark that, subject to appropriate choice principles (which we assume throughout), a compact regular frame *has enough points*, which means that every element is a meet of points. Also, for regular frames, "point" and "maximal element" are synonyms, where the latter is understood to mean maximal strictly below the top. The Lindelöf and the realcompact coreflections of a frame L are denoted by λL and υL, respectively. See [24] for details.

2.2. Rings

An f-ring A is said to have *bounded inversion* if any $a \geq 1$ is a unit in A. If A is an f-ring, we denote its bounded part by A^*. It is not hard to show that, for any $a \in A$, $\frac{a}{1+|a|} \in A^*$. A prime ideal P in a reduced