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Selection properties of texture structures

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ABSTRACT

The aim of this paper is to begin with study of selection principles in texture and ditopological texture spaces and to establish relationships between these selection principles and other covering properties of texture or ditopological spaces. We also investigate the behavior of the properties under consideration under standard operations with ditopological spaces.

1. Introduction

Three classical selection principles are defined in a general form as follows.

Let $\mathcal{A}$ and $\mathcal{B}$ be sets consist of families of subsets of an infinite set X. Then:

$\mathsf{S}_1(\mathcal{A},\mathcal{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

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$\mathsf{U}_{fin}(\mathcal{A}, \mathcal{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that for each n, B_n is a finite subset of A_n and $\{\cup B_n : n \in \mathbb{N}\} \in \mathcal{B}$.

If $\mathcal{O}$ denotes the family of all open covers of a topological space X, then X has the *Rothberger property* (resp. the *Menger property*) if X satisfies $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ (resp. $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$). Denote by $\mathcal{O}_D$ the family of open sets of X such that $\cup\mathcal{O}_D$ is dense in X. If X satisfies $\mathsf{S}_1(\mathcal{O}, \mathcal{O}_D)$ (resp. $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O}_D)$), then we say that X has the *weak Rothberger property* (resp. the *weak Menger property*).

There are infinite games for two players, ONE and TWO, associated to these selection properties. The game $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$ is defined in this way: ONE and TWO play a round for each $n \in \mathbb{N}$. In the n-th round ONE chooses $A_n \in \mathcal{A}_n$, and TWO responds by a finite set $B_n \subseteq A_n$. TWO wins a play $A_1, B_1; A_2, B_2; \cdots$ if $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$; otherwise ONE wins. Other games are defined similarly.

For more information about selection principles and games in topological spaces see the survey papers [7,10,11], and in bitopological spaces the papers [8,9].

These selection properties and associated games will be considered here in two special structures defined below.

Texture space: ([1]) Let S be a set. We work within a subset $\mathcal{S}$ of the power set $\mathcal{P}(S)$ called a *texturing*. A *texturing* of S is a point-separating, complete, completely distributive lattice $(\mathcal{S}, \subseteq) \subseteq \mathcal{P}(S)$, which contains S and $\emptyset$, and for which arbitrary meets coincide with intersections, and finite joins with unions. If $\mathcal{S}$ is a texturing of S, the pair $(S, \mathcal{S})$ is called a *texture*. Throughout the paper we denote by $\bigcap$ and $\bigvee$ meets and joins in a texture $(S, \mathcal{S})$.

For $s \in S$ the sets

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\} \text{ and } Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}$$

are called respectively, the *p-sets* and *q-sets* of $(S, \mathcal{S})$. These sets are used in the definition of many textural concepts.

In a texture, arbitrary joins need not coincide with unions, and clearly, this will be so if and only if $\mathcal{S}$ is closed under arbitrary unions, or equivalently if $P_s \nsubseteq Q_s$ for all $s \in S$. In this case $(S, \mathcal{S})$ is said to be *plain*.

Complementation: ([1]) A mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A$, $\forall A \in \mathcal{S}$ and $A \subseteq B \Longrightarrow \sigma(B) \subseteq \sigma(A)$, $\forall A, B \in \mathcal{S}$ is called a *complementation* on $(S, \mathcal{S})$ and $(S, \mathcal{S}, \sigma)$ is then said to be a *complemented texture*.

If $\mathcal{F}$ is a subfamily of $\mathcal{S}$, then $\sigma(\mathcal{F})$ denotes the set $\{\sigma(F) : F \in \mathcal{F}\}$.

Examples:

(1) For any set X, $(X, \mathcal{P}(X), \pi_X)$ is the complemented *discrete texture* representing the usual set structure of X. Here the complementation $\pi_X(Y) = X \setminus Y$, $Y \subseteq X$, is the usual set complement. Clearly, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$ for all $x \in X$.
(2) For $\mathbb{I} = [0, 1]$ define $\mathcal{I} = \{[0, t] \mid t \in [0, 1]\} \cup \{[0, t) \mid t \in [0, 1]\}$. $(\mathbb{I}, \mathcal{I}, \iota)$ is a complemented texture, which we will refer to as the *unit interval texture*. Here $P_t = [0, t]$ and $Q_t = [0, t)$ for all $t \in I$.
(3) The texture $(\mathbb{L}, \mathcal{L})$ is defined by $\mathbb{L} = (0, 1]$ and $\mathcal{L} = \{(0, r] \mid r \in [0, 1]\}$. For $r \in \mathbb{L}$ $P_r = (0, r] = Q_r$.

Ditopology: A *dichotomous topology* on $(S, \mathcal{S})$ or *ditopology* for short, is a pair (τ, κ) of generally unrelated subsets τ, κ of $\mathcal{S}$ satisfying

(τ_1) $S, \emptyset \in \tau$,
(τ_2) $G_1, G_2 \in \tau \Longrightarrow G_1 \cap G_2 \in \tau$,