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Applications of the reflection functors in paratopological groups

M. Tkachenko [1]

Departamento de Matemáticas, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco 186, Col. Vicentina, C.P. 09340, Del. Iztapalapa, D.F., Mexico

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ABSTRACT

We apply T_i-reflections for $i = 0, 1, 2, 3$, as well as the regular reflection defined by the author in [20] for the further study of paratopological and semitopological groups. We show that many topological properties are invariant and/or inverse invariant under taking T_i-reflections in paratopological groups. Using this technique, we prove that *every* σ-compact paratopological group has the Knaster property and, hence, is of countable cellularity.

We also prove that an arbitrary product of locally feebly compact paratopological groups is a Moscow space, thus generalizing a similar fact established earlier for products of feebly compact topological groups. The proof of the latter result is based on the fact that the functor T_2 of Hausdorff reflection 'commutes' with arbitrary products of semitopological groups. In fact, we show that the functors T_0 and T_1 also commute with products of semitopological groups, while the functors T_3 and *Reg* commute with products of paratopological groups.

1. Introduction

Many results concerning paratopological or semitopological groups have been proved assuming certain separation axioms (see [1,3–5,9,13,19]). This is of course not surprising since neither of the implications

$$T_0 \Longrightarrow T_1 \Longrightarrow T_2 \Longrightarrow \text{regular}$$

is valid in the class of paratopological groups, while the validity of the implication

$$\text{regular} \Longrightarrow \text{Tychonoff}$$

has very recently been proved by T. Banakh and A. Ravsky in [6].

E-mail address: mich@xanum.uam.mx.

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Ravsky [10] started a revision of the use of separation axioms in the proofs involving paratopological groups. He established, in particular, that *every* locally compact paratopological group was a topological group, without assuming any separation axiom (see [10, Proposition 5.5]). Romaguera and Sanchis [12] contributed to this revision, while Ravsky exhibited in [11] more cases where a custom use of separation axioms had been excessive.

In [20,21] we present a 'universal' approach to analyzing the interaction between many topological properties on the one hand and the usual axioms of separation in the categories of semitopological and paratopological groups on the other hand. It is proved in [20, Proposition 2.5] that for every semitopological group G and every separation axiom T_i with $i \in \{0, 1, 2, 3, r, t\}$, there exists a unique T_i-*reflection* of G, i.e. a semitopological group $T_i(G)$ satisfying the T_i separation axiom which maintains a maximal possible amount of information about the original group G. To unify our notation, we use T_r and T_t to refer to the regular and Tychonoff separation axiom, respectively. More precisely, the T_i-reflection of G is a pair $(T_i(G), \varphi_{G,i})$, where $\varphi_{G,i}$ is a continuous homomorphism of G onto a semitopological group $T_i(G)$ satisfying the T_i separation axiom and having the following important property: *Given a continuous mapping $f: G \to X$ to a T_i-space X, there exists a continuous mapping $g: T_i(G) \to X$ such that $f = g \circ \varphi_{G,i}$.*

Informally speaking, the *canonical homomorphism* $\varphi_{G,i}$ is a right divisor for every continuous mapping of G to a T_i-space X. Hence every continuous mapping of G to a T_i-space X can be reconstructed in a canonical way from a continuous mapping of $T_i(G)$ to X. In fact, the T_i-reflection gives rise to a covariant functor 'T_i' from the category of semitopological groups to the category of semitopological groups satisfying the T_i separation axiom (see [20, Corollary 2.8]). The morphisms in both categories are continuous homomorphisms. It is also shown in [20] (see also the introduction in [21]) that the functors T_i with $i \in \{0, 1, 2, 3, r\}$ preserve paratopological groups, i.e. $T_i(G)$ is a paratopological group if so is G. The same conclusion is valid for the functor T_t since every regular paratopological group is Tychonoff [6]. In fact, the functors T_r and T_t coincide in the category of paratopological groups.

In [20,21] we established a number of properties of the canonical homomorphisms $\varphi_{G,i}$ and the functors T_i, for $i \in \{0, 1, 2, 3, r\}$. Here we continue the study in this direction and show in Corollary 3.2 that all the functors T_i's *respect* open subgroups of semitopological groups. We also prove in Section 3 that the functors T_0, T_1, and T_2 commute with products of semitopological groups, while T_3 and $Reg = T_r$ commute with products of paratopological groups.

The reflection functors T_i's are quite useful for the study of semitopological and, especially, paratopological groups. Using Proposition 2.5 of [20], the authors of [23] prove that all concepts of $\mathbb{R}_i$-factorizability, for $i = 0, 1, 2, 3$, coincide in paratopological groups. In other words, similarly to the case of topological groups, $\mathbb{R}$-factorizability (which refers to a special property of continuous real-valued functions) in paratopological groups does not depend on the separation axioms, even if the axioms appear explicitly in the definition of the concept of $\mathbb{R}_i$-factorizability.

Here we present more examples of eliminating the separation axioms. According to [3, Corollary 5.7.12], every Hausdorff σ-compact paratopological group has countable cellularity, which extends the corresponding result established in [17] for topological groups. In fact, almost the same argument shows that the conclusion remains valid for σ-compact paratopological groups satisfying the T_1 separation axiom [19, Theorem 6.12]. Here, in Corollary 2.3, we do the final step by dropping the T_1 separation restriction: *Every σ-compact paratopological group has countable cellularity.* Our argument is based on the fact that the functor T_2 does not change the cellularity of paratopological groups. In turn, the latter fact leans on a special property of the canonical homomorphism $\varphi_{G,2}$, for a paratopological group G.

Another example of elimination of separation restrictions is given in Section 3. It is known that an arbitrary product of locally feebly compact Hausdorff paratopological groups is a Moscow space (see [19, Proposition 7.4]). We show in Theorem 3.7 that the Hausdorff separation requirement can be dropped in this result, so any product of locally feebly compact paratopological groups is a Moscow space.