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Cellularity in subgroups of paratopological groups

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ABSTRACT

It is known that the cellularity of every σ-compact paratopological group is countable, without assuming any separation restrictions on the group. We prove that every subgroup of a σ-compact T_1 paratopological group has countable cellularity, but this conclusion fails for subgroups of σ-compact T_0 paratopological groups. For every infinite cardinal κ, we present a σ-compact subsemigroup H of a Hausdorff topological group such that the cellularity of H equals κ.

We also prove that if S is a non-empty subsemigroup of a topologically periodic semitopological group G, then the closure of S is a subgroup of G. This implies, in particular, that the closure of every non-empty subsemigroup of a precompact topological group G is a subgroup of G and that every subsemigroup of G has countable cellularity.

1. Introduction

In this article we consider several properties of subgroups of σ-compact and precompact paratopological groups. Our main concern is the cellularity of subgroups. It is shown in [9, Corollary 2.3] that the cellularity of every σ-compact paratopological group is countable (no separation restrictions are imposed on the groups). It is also known that every precompact paratopological group has countable cellularity [3, Corollary 3]. In Proposition 2.1 we extend the result from [9] to arbitrary subgroups of σ-compact paratopological groups satisfying the T_1 separation axiom.

The situation changes if one considers subgroups of precompact or σ-compact paratopological groups satisfying the T_0 separation axiom. Indeed, let $\mathbb{T}_S$ be the circle group endowed with the Sorgenfrey topology.

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Then $\mathbb{T}_S$ and its square are precompact Tychonoff paratopological groups, but the "second" diagonal in $\mathbb{T}_S^2$ is an uncountable closed discrete subgroup. It also follows from [4, Corollary 5] that every infinite discrete Abelian group admits a topological isomorphism onto a subgroup of a Hausdorff precompact Abelian group. Therefore, there is no upper bound for the cellularity of subgroups of precompact Hausdorff paratopological groups. Clearly this is impossible in the class of topological groups since every subgroup of a precompact topological group is precompact and, hence, has countable cellularity.

We show in Example 2.2 that there exists a 'simple' σ-compact T_0 paratopological group which contains a discrete subgroup of cardinality 2^ω. In Corollary 2.5 we improve the above conclusion and show that for every infinite cardinal κ, there exists a precompact, σ-compact paratopological group satisfying the T_0 separation axiom which contains a discrete subgroup of cardinality κ.

In Section 3 we study the properties of subsemigroups of topological and paratopological groups. We solve Problem 5.3.5 from [2] in the negative by showing that a compactly generated subsemigroup of a topological group can have an arbitrarily big cellularity. In Theorem 3.3 we prove that if S is a non-empty subsemigroup of a topologically periodic semitopological group G, then the closure of S is a subgroup of G. This implies, in particular, that the closure of every non-empty subsemigroup of a precompact topological group G is a subgroup of G and that every subsemigroup of G has countable cellularity (see Corollaries 3.4 and 3.6).

Finally in Proposition 3.8 we establish that every feebly compact *almost topological* group is a topological group.

1.1. Notation and terminology

A *paratopological* (*semitopological*) group is a group with a topology such that multiplication on the group is jointly (separately) continuous. If τ is the topology of a paratopological group G, then the family

$$\tau^{-1} = \{U^{-1} : U \in \tau\}$$

is also a topology on G and $G' = (G, \tau^{-1})$ is again a paratopological group *conjugated* to G. It is clear that the inversion on G is a homeomorphism of G onto G'. The upper bound $\tau^* = \tau \vee \tau^{-1}$ is a topological group topology on G and $G^* = (G, \tau^*)$ is a topological group *associated* to G. If G is a T_0-space, then the topological group G^* is Hausdorff.

A non-empty subset S of a group G is a *subsemigroup* of G if $xy \in S$ for all $x, y \in S$.

The cellularity of a space X is denoted by $c(X)$. The cardinality of the continuum is $\mathfrak{c}$, i.e., $\mathfrak{c} = 2^\omega$. The set of positive integers in $\mathbb{N}^+$.

2. Subgroups of σ-compact paratopological groups

According to Corollary 2.3 of [9], the cellularity of every σ-compact paratopological group is countable. This result suggests the natural question about the cellularity of arbitrary subgroups of σ-compact paratopological groups. Quite unexpectedly, the answer to this question depends on axioms of separation. We start with the case of σ-compact paratopological groups satisfying the T_1 separation axiom.

Proposition 2.1. *Suppose that H is a subgroup of a σ-compact paratopological group G satisfying the T_1 separation axiom. Then the cellularity of H is countable.*

Proof. Let $i_G: G^* \to G$ be the identity mapping of the topological group G^* associated to G onto G. By [1, Lemma 2.2], the group G^* is topologically isomorphic to the diagonal $\Delta = \{(x, x) : x \in G\}$ of the product $G \times G'$, where G' is the paratopological group conjugated to G, and Δ is a closed subgroup of $G \times G'$. Since