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Asymmetric norms, cones and partial orders

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ABSTRACT

If X is a real vector space and p an asymmetric norm on X, the set $C_p = \{x \in X : p(-x) = 0\}$ is a proper cone in X which induces a partial order on X compatible with the linear structure of X. Using the norm $p^s(x) = \max\{p(x), p(-x)\}$, a second asymmetric norm can be defined by $q_p(x) = \inf\{p^s(x + y) : y \in C_p\}$. In the case where the partial order induced by C_p is a lattice order, it is possible to define a third asymmetric norm by $p^+(x) = p(x^+)$, where x^+ is the positive part of x. The paper investigates the relationships between these three asymmetric norms, with special attention to the case where X is finite-dimensional.

1. Introduction

There is a close relationship between asymmetric norms on real vector spaces and cones in such spaces. If p is an asymmetric norm on the real vector space X, then $C_p = \{x \in X : p(-x) = 0\}$ is a proper cone in X which induces a partial order on X that is compatible with the linear operations. The symmetrization p^s of p is a norm on X, and the cone C_p is closed in the topology induced by p^s.

On the other hand, if C is a proper closed cone in a real normed vector space X, it is possible to define an asymmetric norm q_C on X in terms of the cone C and the norm, and $C_{q_C} = \{x \in X : q_C(-x) = 0\} = C$. If the partial order on X induced by the cone C is a lattice order, it is possible to define a second asymmetric norm q^+ on X using this lattice structure and the norm, and as before $C_{q^+} = \{x \in X : q^+(-x) = 0\} = C$. These two are arguably the best known examples of asymmetric norms (see, for example, [3] and [5]).

Starting with an asymmetric norm p on a real vector space X, it is therefore possible to use the cone C_p and the norm p^s to define two further asymmetric norms q_{C_p} and p^+ (the latter in the case where

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C_p induces a lattice order) on X. The aim of this paper is to investigate the relationships between these three asymmetric norms and the topologies they generate. In the process the link between convex absorbent sets and asymmetric norms is investigated, and a number of examples of asymmetric norms constructed to illustrate these relationships. Particular attention is given to the situation in finite-dimensional vector spaces.

We introduce the terminology and notation we will use in what follows. The set of non-negative real numbers will be denoted by $\mathbb{R}^+$, and X will always denote a real vector space. A function $p : X \to \mathbb{R}^+$ will be called an *asymmetric seminorm* on X if for all $x, y \in X$ and $\lambda \in \mathbb{R}^+$,

(a) $p(\lambda x) = \lambda p(x)$;
(b) $p(x + y) \leq p(x) + p(y)$.

If in addition we also have

(c) $p(x) = p(-x) = 0$ if and only if $x = 0$,

p will be called an *asymmetric norm*, and the pair (X, p) an *asymmetrically normed space*.

If p is an asymmetric norm on X, the function $p^{-1} : X \to \mathbb{R}^+$ defined by

$$p^{-1} = p(-x), \quad x \in X$$

is also an asymmetric norm, the asymmetric norm conjugate to p.

The symmetrization of the asymmetric norm p is the function $p^s : X \to \mathbb{R}^+$ given by

$$p^s(x) = \max\{p(x), p(-x)\}, \quad x \in X$$

and this is easily seen to be a norm on X.

An asymmetric norm p induces a quasi-metric d_p on X defined by

$$d_p(x, y) = p(y - x) \text{ for all } x, y \in X.$$

For $x \in X, r > 0$ we define the balls

$$B_r^p(x) = \{y \in X : d_p(x, y) < r\} = \{y \in X : p(y - x) < r\} \text{ and}$$
$$B_r^p[x] = \{y \in X : d_p(x, y) \leq r\} = \{y \in X : p(y - x) \leq r\}.$$

The family $\{B_r^p(x) : r > 0\}$ forms a fundamental system of neighborhoods for x for a T_0 topology τ_p on X, which we shall refer to as the topology induced by p. The Hausdorff topology τ_{p^s} induced by the norm p^s is clearly finer than the topologies τ_p and $\tau_{p^{-1}}$. The balls $B_r^p[x]$ are p^{-1}-closed, and therefore also p^s-closed, as is easily checked. Note also that since $p(x) \leq p^s(x)$ for every $x \in X$, it follows that p is τ_{p^s}-continuous.

As a simple but important example we mention the asymmetric norm p_1 on $\mathbb{R}$ (regarded as a real vector space) defined for all $x \in \mathbb{R}$ by $p_1(x) = x^+$, where $x^+ = x \vee 0 = \max\{x, 0\}$ is the positive part of x. In this case $p_1^{-1}(x) = \max\{-x, 0\} = x^-$ (the negative part of x), and $p_1^s(x) = \max\{x^+, x^-\} = |x|$.

Two asymmetric norms p and q on X are equivalent if they induce the same topology on X. This will be the case if and only if there are positive constants C_1 and C_2 such that for all $x \in X$, $C_1 p(x) \leq q(x) \leq C_2 p(x)$.

For further information on asymmetric norms we refer the reader to the monograph [4].

In the next section we look at the close relationship between asymmetric norms, convex absorbent sets and cones and introduce the asymmetric norm q_C. In Section 3 the partial order induced by a cone and the