Topology and its Applications 193 (2015) 182–191

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Locally σ-compact rectifiable spaces ☆

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A R T I C L E I N F O

Article history:
Received 16 July 2015
Accepted 21 July 2015
Available online 29 July 2015

MSC:
22A05
54E20
54E35
54H11

Keywords:
Rectifiable spaces
Locally σ-compact
Locally compact
Paracompact
Zero-dimension
k_ω-Spaces
Rectifiable complete spaces

A B S T R A C T

A topological space G is said to be a *rectifiable space* provided that there are a surjective homeomorphism $\varphi : G \times G \to G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$, $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \to G$ is the projection to the first coordinate. In this paper, we first prove that each locally compact rectifiable space is paracompact, which gives an affirmative answer to Arhangel'skii and Choban's question (Arhangel'skii and Choban [2]). Then we prove that every locally σ-compact rectifiable space with a *bc*-base is locally compact or zero-dimensional, which improves Arhangel'skii and van Mill's result (Arhangel'skii and van Mill [4]). Finally, we prove that each k_ω-rectifiable space is rectifiable complete.

1. Introduction

Recall that a *paratopological group* G is a group G with a topology such that the product mapping of $G \times G$ into G is jointly continuous. The space G is called a *topological group* if it is a paratopological group and the inverse mapping of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous. In addition, the space G is said to be a *rectifiable space* provided that there are a surjective homeomorphism $\varphi : G \times G \to G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$, $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \to G$ is the projection to the first coordinate. If G is a rectifiable space, then φ is called a *rectification* on G. It is well known that rectifiable spaces and paratopological groups are two good generalizations of topological groups. In fact, for a topological group with the neutral element e, it is easy to see that the mapping $\varphi(x, y) = (x, x^{-1}y)$ is a rectification on G. However, the 7-dimensional sphere S_7

☆ Supported by the NSFC (Nos. 11201414, 11471153), the Natural Science Foundation of Fujian Province (No. 2012J05013) of China and Training Programme Foundation for Excellent Youth Researching Talents of Fujian's Universities (JA13190).

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http://dx.doi.org/10.1016/j.topol.2015.07.008
0166-8641/

is rectifiable but not a topological group [16, §3]. The Sorgenfrey line G is a paratopological group with no rectification on G. Further, it is easy to see that each rectifiable space is homogeneous. Recently, the study of rectifiable spaces has become an interesting topic in topological algebra, see [1,2,7–13]. In particular, the following theorem plays an important role in the study of rectifiable spaces.

Theorem 1.1. *([5,7,15]) A topological space G is rectifiable if and only if there exist an element $e \in G$ and two continuous mappings $p : G^2 \to G$, $q : G^2 \to G$ such that for any $x, y \in G$ the next identities hold:*

$$p(x, q(x, y)) = q(x, p(x, y)) = y \text{ and } q(x, x) = e.$$

Given a rectification φ of the rectifiable space G, we can obtain the mappings p and q in Theorem 1.1 as follows. Let $p = \pi_2 \circ \varphi^{-1}$ and $q = \pi_2 \circ \varphi$. Then the mappings p and q satisfy the identities in Theorem 1.1, and both are open mappings.

Let G be a rectifiable space, and let p be the multiplication on G. Then we sometimes write $x \cdot y$ instead of $p(x, y)$ and $A \cdot B$ instead of $p(A, B)$ for any $A, B \subset G$. Obviously, $q(x, y)$ is an element in G such that $x \cdot q(x, y) = y$. Moreover, since $x \cdot e = x \cdot q(x, x) = x$ and $x \cdot q(x, e) = e$, it follows that e is a right neutral element for G and $q(x, e)$ is a right inverse for x. Therefore, a rectifiable space G is a topological algebraic system with operations p and q, a 0-ary operation e, and identities as above. Further, the multiplication operation p of this algebraic system need not satisfy the associative law. Note that every topological loop is rectifiable.

In this paper, we mainly discuss the topological properties of locally σ-compact rectifiable spaces. The paper is organized as follows:

In Section 2, we mainly prove that each locally compact rectifiable space is paracompact, which gives an affirmative answer to a question of A.V. Arhangel'skii and M.M. Choban.

In Section 3, we mainly prove that every locally σ-compact rectifiable space with a bc-base is locally compact or zero-dimensional, which improves a result of A.V. Arhangel'skii and J. van Mill.

In Section 4, we mainly prove that each k_ω-rectifiable space is rectifiable complete.

All the spaces considered in this paper are supposed to be Hausdorff unless stated otherwise. The notation ω denotes the first countable infinite order. The letter e denotes the right neutral element of a rectifiable space. For undefined notation and terminologies, the reader may refer to [3] and [6].

2. Locally compact rectifiable spaces

In this section, we shall discuss the locally σ-compact rectifiable spaces, and show that each locally σ-compact rectifiable space is paracompact.

In [2], A.V. Arhangel'skii and M.M. Choban posed the following two questions:

Question 2.1. *([2, Problem 5.9]) Is every rectifiable p-space paracompact? What if the space is locally compact?*

Question 2.2. *([2, Problem 5.10]) Is every rectifiable p-space a D-space?*

It is well known that a paracompact p-space is a D-space. Therefore, if we can prove each rectifiable p-space is paracompact, then the answer to Question 2.2 is also positive. However, we will prove that each locally compact rectifiable space is paracompact, which gives an affirmative answer to the second part of Question 2.1, see Corollary 2.11. By this result, we give a partial answer to Question 2.2, see Corollary 2.14. First of all, we give some concepts and technical lemmas.

A space X is called *locally compact* (resp. *locally σ-compact*) if, for each $x \in X$, there exists a neighborhood U at x in X such that the closure of U in X is compact (resp. σ-compact).