Topology and its Applications 193 (2015) 206–216

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Variations of consonance and the rational numbers

Francis Jordan [1]

Department of Mathematics and Computer Science, Queensborough Community College, Bayside, NY 11364, USA

ARTICLE INFO

Article history:
Received 14 January 2015
Received in revised form 7 July 2015
Accepted 13 July 2015
Available online 29 July 2015

MSC:
primary 54A20
secondary 54C35, 03E75

Keywords:
Upper-Kuratowski topology
Co-compact topology
Consonant
Rational
Fin-trivial
Splitting topology

ABSTRACT

It is a result of A. Bouziad that every regular, first countable, totally imperfect space with no isolated points is not $\boldsymbol{Fin}$-trivial. We prove that every regular totally imperfect space containing a copy of the rational numbers is not $\boldsymbol{Fin}$-trivial in a strong sense. Our result generalizes that of Bouziad to a larger class of spaces and gives a strengthened conclusion. As a corollary we conclude that various splitting topologies on the space of continuous real-valued functions defined on a metric space need not coincide.

1. Introduction

We denote the collections of closed subsets, compact subsets, and finite subsets of a topological space X by $\boldsymbol{F}(X)$, $\boldsymbol{K}(X)$, and $\boldsymbol{Fin}(X)$, respectively. When the space X is understood we will often drop the symbol (X) and write, for example, $\boldsymbol{Fin}$ instead of $\boldsymbol{Fin}(X)$. Two well studied topologies on $\boldsymbol{F}$ are the upper-Kuratowski topology and the compact-open topology, which we denote by $T(uk)$ and τ_{co}, respectively. Let $\boldsymbol{H} \subseteq \boldsymbol{F}$. A space is called $\boldsymbol{H}$-trivial [16] provided that $T(uk)|\boldsymbol{H} = \tau_{\text{co}}|\boldsymbol{H}$, where $T(uk)|\boldsymbol{H}$ and $\tau_{\text{co}}|\boldsymbol{H}$ denote the subspace topologies on $\boldsymbol{H}$ induced by $T(uk)$ and τ_{co} respectively. The $\boldsymbol{F}$-trivial spaces are also known as *consonant* spaces, which is the name given in [7] where the study of the coincidence of $T(uk)$ and τ_{co} for general topological spaces is initiated. A topological space is said to be *perfect* provided that it is nonempty, Hausdorff, compact, and has no isolated points. The space X is called *totally imperfect* provided that it has no perfect subspace. Using the fact that compact Hausdorff spaces are regular, one can use a

E-mail address: fejord@hotmail.com.

[1] Support for this project was provided by a PSC-CUNY award jointly provided by The Professional Staff Congress and The City University of New York TRADA 66029-00 44.

http://dx.doi.org/10.1016/j.topol.2015.07.007
0166-8641/

standard Cantor tree construction to show that every perfect space has cardinality at least that of the real numbers. By ω we mean the set of non-negative integers.

In [16] the first non-consonant metric space was constructed and it was asked whether there is a non-consonant separable metric space. Examples of non-consonant separable metric spaces and first countable spaces, in particular the rational numbers $\mathbb{Q}$ with their usual topology, soon followed [1,4,3], and [5]. The result of greatest generality in this direction is:

Proposition 1. *([3, Corollary 3.8]) Every regular first countable space without isolated points, all compact subsets of which are countable, is not* $\boldsymbol{Fin}$*-trivial (and hence, non-consonant).*

It should be noted that even though regularity is not explicitly mentioned in the result of Bouziad, all spaces in the section of the paper where the result appears are assumed to be regular. Indeed, the result of Bouziad relies on the theory of Prohorov spaces which are sometimes (see, for example, the survey [17]) assumed to be regular. The author has attempted with no success to determine whether the assumption of Hausdorffness is sufficient for Bouzaid's result. In [13] and [9] two notions weaker than consonance called infraconsonant and compact-family-splittable, respectively, are considered. The notions of infraconsonant and compact-family-splittable are closely tied to the joint continuity and separate continuity of the group operations on $C(X)$ with the Isbell topology, respectively. The following is a corollary of Theorem 14 of this paper and improves Proposition 1:

Corollary 2. *If X is a regular totally imperfect space with no isolated points that contains a copy of $\mathbb{Q}$, then X is neither* $\boldsymbol{Fin}$*-trivial nor compact-family-splittable.*

We now show how Proposition 1 follows from Corollary 2. Suppose X satisfies the hypothesis of Proposition 1. It is enough to show that X also satisfies the hypothesis of Corollary 2. It is a routine exercise to verify that since X is regular and first countable with no nonisolated points, then X must contain a copy of $\mathbb{Q}$. It remains to show that X is totally imperfect. Suppose $A \subseteq X$ is compact. Since A is countable and every perfect space is uncountable, A must have an isolated point. Thus, X is totally imperfect.

We give an example of space that satisfies the hypothesis of Corollary 2, but not the hypothesis of Proposition 1. Let $\{B_n\}_{n\in\omega}$ be a countable base of nonempty open sets for $\mathbb{Q}$. Let $\beta(\mathbb{Q})$ stand for the Stone–Cech compactification of $\mathbb{Q}$. For each $n \in \omega$ pick $x_n \in \beta(\mathbb{Q}) \setminus \mathbb{Q}$ so that $x_n \in \mathrm{cl}_{\beta(\mathbb{Q})}(B_n)$. Let $X = \mathbb{Q} \cup \{x_n : n \in \omega\}$. Since $\beta(\mathbb{Q})$ is compact and Hausdorff, X is completely regular. Since X is countable and every perfect space is uncountable, X is totally imperfect. So, X satisfies the hypothesis of Corollary 2. However, X is not first countable at any point in $X \setminus \mathbb{Q}$. By way of contradiction, assume that $x \in X \setminus \mathbb{Q}$ has countable local base of open sets. Since the rationals are dense in X, there is sequence $(q_n)_{n=1}^{\infty}$ on $\mathbb{Q}$ such that $\lim q_n = x$. Since $\{q_n : n \in \omega\}$ is closed and discrete in $\mathbb{Q}$, there is a continuous bounded function $f: \mathbb{Q} \to \mathbb{R}$ so that $f(q_n) = 0$ when n is even and $f(q_n) = 1$ when n is odd. Any continuous bounded function $\mathbb{Q}$ into $\mathbb{R}$ has a bounded continuous extension to $\beta(\mathbb{Q})$, and hence X. So, $\lim (f^*(q_n))_{n\in\omega} = f^*(x)$, where f^* is the continuous extension of f, which is impossible by the way we defined f. Finally, notice that $\mathbb{Q}$ is neither open nor closed in X. So, Proposition 1 cannot be combined with any of the usual theorems about open or closed subspaces of consonant or $\boldsymbol{Fin}$-trivial spaces to conclude that X is not $\boldsymbol{Fin}$-trivial or even non-consonant.

2. Preliminaries

All filters considered are nondegenerate, that is, they do not contain the empty set. Given two families of sets $\mathcal{F}$ and $\mathcal{G}$ we say that $\mathcal{F}$ is *coarser* than $\mathcal{G}$ (or, $\mathcal{G}$ is *finer* than $\mathcal{F}$) and write $\mathcal{F} \leq \mathcal{G}$ provided that for every $F \in \mathcal{F}$ there is a $G \in \mathcal{G}$ such that $G \subseteq F$, we will apply this notation to filters and topologies. For a