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Knot projections with reductivity two

Noboru Ito [a,*], Yusuke Takimura [b]

[a] Waseda Institute for Advanced Study, 1-6-1 Nishi Waseda, Shinjuku-ku, Tokyo 169-8050, Japan
[b] Gakushuin Boys' Junior High School, 1-5-1 Mejiro, Toshima-ku, Tokyo 171-0031, Japan

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ABSTRACT

Reductivity of knot projections refers to the minimum number of splices of double points needed to obtain reducible knot projections. Considering the type and method of splicing (Seifert type splice or non-Seifert type splice, recursively or simultaneously), we can obtain four reductivities containing Shimizu's reductivity, three of which are new. In this paper, we determine knot projections with reductivity two for all four of the definitions. We also provide easily calculated lower bounds for some reductivities. Further, we detail properties of each reductivity, and describe relationships among the four reductivities with examples.

1. Introduction

A *knot projection* is the image of a generic immersion of an unoriented circle into the sphere S^2. By this definition, a knot projection is an unoriented generic immersed curve on S^2, also called a *spherical curve*. In this study, we assume that every knot projection has at least one double point. As previously proven, every knot projection without 1- and 2-gons has 3-gons. As shown in Fig. 1, there are four types of 3-gons. Of interest is the still-unanswered question below [4, Question 3.2].

Question 1. (Shimizu [4]) Is it true that every prime knot projection without 1- and 2-gons has at least one 3-gon of type A, B, or C, as shown in Fig. 1?

If it is true, we can more easily prove the triple chord's theorem [3]:

Fact 1. *(Ito-Takimura [3]) Every chord diagram of a prime knot projection without* 1- *and* 2-*gons contains* $\otimes$.

* Corresponding author.
E-mail addresses: noboru@moegi.waseda.jp (N. Ito), Yusuke.Takimura@gakushuin.ac.jp (Y. Takimura).

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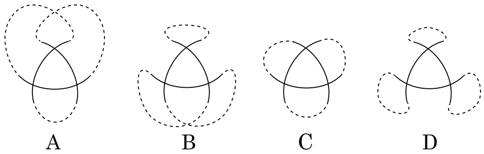

Fig. 1. All types of 3-gons. For each 3-gon, three dotted arcs show the connections of branches of three double points.

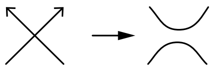

Fig. 2. Non-Seifert splice A^{-1}.

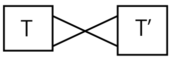

Fig. 3. A nugatory crossing between two (1, 1)-tangles, T and T'.

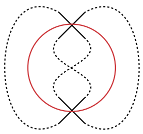

Fig. 4. A simple closed curve and a type of knot projection. Dotted arcs show the connections of branches between two double points.

Here, a chord diagram is a circle, on which paired points are placed such that each pair of two points, corresponding to two pre-images of a double point of a knot projection, is connected by a chord.

Shimizu's reductivity $r(P)$ [4], closely related to Question 1, is defined as follows: Shimizu's reductivity $r(P)$ of a knot projection P is the minimum number of non-Seifert splices defined by Fig. 2, applied recursively, to obtain a *reducible knot projection* from P. Here, a reducible knot projection P is a knot projection having a *nugatory crossing*, as shown in Fig. 3. A non-reducible knot projection is called a *reduced knot projection*. Traditionally, an arbitrary non-Seifert splice is often denoted by A^{-1} [1]. Note that A^{-1} does not depend on the selection of the orientation of a knot projection. Shimizu's Question 2 remains unanswered, as well.

Question 2. (Shimizu [4]) Is it true that $r(P) \leq 3$ for every knot projection P?

As Shimizu points out, if the answer to Question 1 is *yes*, the answer to Question 2 is also *yes* [4].

In general, it is not easy to compute Shimizu's reductivity r. However, this paper provides the lower bounds that can be easily calculated for an infinite family of knot projections (Theorem 4 and Examples 1 and 2). Moreover, we explicitly obtain the necessary and sufficient condition for a knot projection to satisfy $r(P) = 2$ (Theorem 1).

As a first step, Ito and Takimura [3] establish the necessary and sufficient condition that a knot projection has $r(P) = 1$ (Fact 2).

Fact 2. *Let P be a reduced knot projection P. There exists a circle intersecting P at just two double points of P, as shown in Fig. 4, if and only if $r(P) = 1$.*

As a corollary,

Fact 3. *(Ito–Takimura [3]) Every chord diagram of a knot projection P with $r(P) = 1$ has a sub-chord diagram* .