Topology and its Applications 193 (2015) 302–308

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

A note on Mazur type Stein fillings of planar contact manifolds

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ARTICLE INFO

Article history:
Received 19 June 2014
Accepted 21 July 2015
Available online 6 August 2015

MSC:
57R65
57R17
57M27

Keywords:
Contractible bounds
Contact manifolds
Casson invariant

ABSTRACT

We construct a family of Stein fillable contact homology 3-spheres $\{(M_n, \xi_n)\}_{n\geq 1}$ such that the contact structure ξ_n is supported by an open book with planar page, and a Stein filling of (M_n, ξ_n) is of Mazur type for each n.

1. Introduction

Let M be a closed, oriented, connected 3-manifold. A 2-plane field ξ on M is called a *contact structure* on M if it is represented as $\xi = \ker \alpha$ for some 1-form α on M satisfying $\alpha \wedge d\alpha > 0$. An open book of M is called a *supporting open book* for $\xi = \ker \alpha$ if $d\alpha$ is an area form of each page and $\alpha > 0$ on its binding. Giroux [10] (cf. [6]) showed that there is a one-to-one correspondence between contact structures on M up to isotopy and open books of M up to an equivalence called *positive stabilization*.

Using this correspondence, Etnyre and Ozbagci [7] introduced the following invariants of contact structures by their supporting open books. For a contact 3-manifold (M, ξ), the *support genus* $\mathrm{sg}(\xi)$ of ξ is the minimal genus of a page of a supporting open book for ξ, and the *binding number* $\mathrm{bn}(\xi)$ of ξ is the minimal number of binding components of a supporting open book for ξ which has a page of genus $\mathrm{sg}(\xi)$. They classified the contact structures ξ on M with $\mathrm{sg}(\xi) = 0$ and $\mathrm{bn}(\xi) \leq 2$. Arikan [1] also classified those with $\mathrm{sg}(\xi) = 0$ and $\mathrm{bn}(\xi) = 3$. The standard contact structure ξ_{st} on S^3 is the only Stein fillable contact structure on a homology 3-sphere in their classification. It is well-known after Eliashberg [5] that D^4 is a unique Stein filling of (S^3, ξ_{st}). Therefore it is a natural question whether there exists a homology 3-sphere M admitting a Stein fillable contact structure ξ with $\mathrm{sg}(\xi) = 0$ and $\mathrm{bn}(\xi) = 4$ and what is a Stein filling of (M, ξ).

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http://dx.doi.org/10.1016/j.topol.2015.07.009
0166-8641/

In this note, we give an answer to this question by constructing a family of Stein fillable contact 3-manifolds with $\mathrm{sg}(\xi) = 0$ and $\mathrm{bn}(\xi) = 4$ whose Stein filling is not diffeomorphic to D^4. Note that in [14] the author proved that any Stein fillable contact homology 3-sphere (M, ξ) with $\mathrm{sg}(\xi) = 0$ and $\mathrm{bn}(\xi) = 4$ admits a unique Stein filling. Thus the main result in this note gives infinitely many examples for which the uniqueness result holds.

Furthermore, these Stein fillings are of *Mazur type*. Mazur [13] introduced a contractible 4-manifold whose boundary is a homology 3-sphere not diffeomorphic to S^3, called the Mazur manifold. A Mazur type manifold was defined as a generalization of the Mazur manifold (for the precise definition, see Definition 3.1).

Now we are ready to state the main theorem in this note.

Theorem 1.1. *There exists a family of contact homology 3-spheres* $\{(M_n, \xi_n)\}_{n\geq 1}$ *such that*

1. $M_1, M_2, \dots$ *are mutually not diffeomorphic,*
2. *each contact structure* ξ_n *is Stein fillable and supported by an open book with page a 4-holed sphere, and*
3. *a Stein filling* X_n *of* (M_n, ξ_n) *is a Mazur type manifold.*

We have one more motivation for our work. Many examples of *corks*, which are compact contractible Stein 4-manifolds admitting a nice involution, are known to be of Mazur type. Some of them work to detect an exotic 4-manifold pair. See [2] and [4]. Thus the Stein fillings of the contact manifolds in the above theorem are candidates for corks, and we may construct an exotic pair.

This note is constructed as follows. In Section 2, we review some definitions and properties of positive Lefschetz fibrations and the Casson invariant. In Section 3, we prove Theorem 1.1. The proof of this theorem is based on works of Loi and Piergallini [12] and Akbulut and Ozbagci [3], who proved that, for any *positive allowable Lefschetz fibration (PALF)* $f : X \to D^2$, there exists a Stein fillable contact structure ξ on $M = \partial X$ such that ξ is supported by the open book obtained from f, and X is a Stein filling of (M, ξ). Thus first we give appropriate ordered collection of mapping classes as a monodromy of each PALF belonging to a family of PALFs. Drawing a Kirby diagram of the PALF and performing Kirby calculus, we examine it. Finally we calculate the Casson invariant of its boundary and finish the proof of Theorem 1.1.

2. Preliminaries

We first review positive Lefschetz fibrations and the Casson invariant. For more about contact topology we refer the reader to [6,9], and [15].

Throughout this note we will work in the smooth category and consider the homology groups with integer coefficients.

2.1. Positive Lefschetz fibrations

Let X be a compact, oriented, smooth 4-manifold and B a compact, oriented, smooth 2-manifold.

Definition 2.1. A map $f : X \to B$ is called a *positive Lefschetz fibration* if there exist points $b_1, b_2, \dots, b_m$ in $\mathrm{Int}(B)$ such that

1. $f|f^{-1}(B - \{b_1, b_2, \dots, b_m\}) : f^{-1}(B - \{b_1, b_2, \dots, b_m\}) \to B - \{b_1, b_2, \dots, b_m\}$ is a fiber bundle over $B - \{b_1, b_2, \dots, b_m\}$ with fiber diffeomorphic to an oriented surface F,
2. $b_1, b_2, \dots, b_m$ are the critical values of f with a unique critical point $p_i \in f^{-1}(b_i)$ of f for each i,
3. for each b_i and p_i, there are local complex coordinate charts with respect to the given orientations of X and B such that locally f can be written as $f(z_1, z_2) = z_1^2 + z_2^2$, and
4. no fiber contains a (-1)-sphere, that is, an embedded sphere with self-intersection number -1.