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Separating sets by cliquish functions

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ABSTRACT

In this paper we characterize the pairs $\langle A_0, A_1 \rangle$ of disjoint subsets of topological space X which can be separated by a cliquish function.

1. Introduction

The problem of separating sets by functions is closely related to the classical Urysohn Lemma which states: if X is a T_4-space and the sets $A_0, A_1 \subset X$ are disjoint and closed, then there exists a continuous function $f: X \to \mathbb{R}$ such that $f = 0$ on A_0 and $f = 1$ on A_1, and if moreover A_0 and A_1 are G_δ sets, then we can require that $f(x) \in (0, 1)$ for each $x \in X \setminus (A_0 \cup A_1)$. In [1] A. Maliszewski replaced the continuity of the function f by Darboux property. More precisely, he examined when, given two sets $A_0, A_1 \subset \mathbb{R}$, we can find a Darboux function f such that $f = 0$ on A_0 and $f = 1$ on A_1. He called this "the classical separation property". Moreover he investigated the pairs $\langle A^-, A^+ \rangle$ for which there exists a Darboux function f such that $f < 0$ on A^- and $f > 0$ on A^+ and he called this "a new separation property". Similar problems for the family of quasi-continuous functions were examined by M. Kowalewski in [2] and for the family Darboux quasi-continuous functions by M. Kowalewski and A. Maliszewski in [3]. In 2013 P. Szczuka solved the problem of separating sets by an upper semicontinuous function and by a Darboux upper semicontinuous function [4]. In this paper we deal with the family of cliquish functions.

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2. Preliminaries

The letters $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and the real line, respectively. Let X be a topological space. For each set $A \subset X$, we use the symbols $\operatorname{cl} A$, $\operatorname{int} A$, and $\operatorname{bd} A$ to denote the closure, the interior, and the boundary of A, respectively. Let $f: X \to \mathbb{R}$. For each $a \in \mathbb{R}$ we define

$$[f = a] \stackrel{\text{df}}{=} \{x \in X\,;\, f(x) = a\}.$$

Similarly we define the sets $[f > a]$ and $[f < a]$. By $\mathcal{C}(f)$ we denote the set of all continuity points of f. Let $A, B \subset X$, $A \cap B \neq \emptyset$ and $f: A \to \mathbb{R}$. The *oscillation* of f on the set B we call the number

$$\omega(f, B) \stackrel{\text{df}}{=} \sup\big\{|f(x) - f(t)|\,;\, x, t \in A \cap B\big\}.$$

The *oscillation* of f at a point $x \in A$ we call the number

$$\omega(f, x) \stackrel{\text{df}}{=} \inf\big\{\omega(f, U)\,;\, U \text{ is open neighborhood of } x\big\}.$$

We say that f is *cliquish* [5] *at a point* $x \in X$, if for each open neighborhood V of x and each $\varepsilon > 0$, there is a nonempty open set $G \subset V$ such that $\omega(f, G) < \varepsilon$. If f is cliquish at each point $x \in X$, then we say that f is *cliquish*. By $\mathcal{C}_q(f)$ we denote the set of all cliquishness points of f.

Theorem 2.1. *([5, Thm. 4]) The points of the discontinuity of every cliquish function constitute a set of Baire's first category.*

3. Classical separation property

Theorem 3.1. *Let A_0 and A_1 be subsets of a topological space X. The following conditions are equivalent:*

i) *there is a cliquish function $f: X \to \mathbb{R}$ such that*

$$A_0 = [f = 0] \qquad \textit{and} \qquad A_1 = [f = 1]; \tag{1}$$

ii) *$A_0 \cap A_1 = \emptyset$, the sets $\operatorname{cl} A_0 \setminus A_0$, $\operatorname{cl} A_1 \setminus A_1$ are meagre and $\operatorname{cl} A_0 \cap \operatorname{cl} A_1$ is nowhere dense in X.*

Proof. i) ⇒ ii). Let $f: X \to \mathbb{R}$ be a cliquish function satisfies (1). Observe that $A_0 = f^{-1}(0)$ and $A_1 = f^{-1}(1)$, so $A_0 \cap A_1 = \emptyset$.

Put $x \in \operatorname{cl} A_0 \setminus A_0$ and define the set $V \stackrel{\text{df}}{=} \mathbb{R} \setminus \{0\}$. Since $x \notin A_0$, so $x \in f^{-1}(V)$. On the other side

$$x \in \operatorname{cl} A_0 \subset \operatorname{cl} f^{-1}(0) = X \setminus \operatorname{int}\big(X \setminus f^{-1}(0)\big) = X \setminus \operatorname{int} f^{-1}(V).$$

So $x \notin \operatorname{int} f^{-1}(V)$ and then $x \notin \mathcal{C}(f)$. This proves that $\operatorname{cl} A_0 \setminus A_0$ is subset of $X \setminus \mathcal{C}(f)$ which is meagre. Similarly we can proof that the set $\operatorname{cl} A_1 \setminus A_1$ is meagre.

Now suppose that there exists $x \in \operatorname{int}(\operatorname{cl} A_0 \cap \operatorname{cl} A_1)$. Since $x \in \mathcal{C}_q(f)$, so there exists nonempty open set $V \subset \operatorname{int}(\operatorname{cl} A_0 \cap \operatorname{cl} A_1)$, such that $\omega(f, V) < 1$. Then

$$V \cap \operatorname{cl} A_0 \neq \emptyset \neq V \cap \operatorname{cl} A_1 \quad \Rightarrow \quad V \cap A_0 \neq \emptyset \neq V \cap A_1.$$

But then $\omega(f, V) \geq 1$ contrary to assumption. So the set $\operatorname{cl} A_0 \cap \operatorname{cl} A_1$ is nowhere dense.