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On the unit of a monoidal model category

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ABSTRACT

In this paper we show how to modify cofibrations in a monoidal model category so that the tensor unit becomes cofibrant while keeping the same weak equivalences. We obtain applications to enriched categories and coloured operads in stable homotopy theory.

A *monoidal model category* is a model category $\mathscr{M}$ with a monoidal structure, consisting of a tensor product $\otimes: \mathscr{M} \times \mathscr{M} \to \mathscr{M}$, a unit $\mathbf{1} \in \operatorname{Ob} \mathscr{M}$, and coherent associativity and unit isomorphisms, such that the following two axioms hold:

- *Push-out product axiom*: Given cofibrations $f: X \to Y$ and $g: U \to V$, their push-out product $f \odot g: X \otimes V \cup_{X \otimes U} Y \otimes U \to Y \otimes V$ is a cofibration. Moreover, if f or g is a trivial cofibration then so is $f \odot g$.
- *Unit axiom*: There exists a cofibrant resolution of the tensor unit $q: \tilde{\mathbf{1}} \xrightarrow{\sim} \mathbf{1}$ (i.e. a weak equivalence with cofibrant source) such that, for any cofibrant object X in $\mathscr{M}$, $q \otimes X$ and $X \otimes q$ are weak equivalences.

This is essentially Hovey's definition [4, §4] with Schwede–Shipley's terminology [15]. It induces a monoidal structure on the homotopy category $\operatorname{Ho} \mathscr{M}$. In the symmetric case, if we want to equip the category of monoids with a transferred model structure, we can include the *monoid axiom* [15, Definition 3.3].

In recent applications, there seems to be a pressing need for a cofibrant tensor unit $\mathbf{1}$, e.g. [1,2,8]. However, examples with non-cofibrant tensor units, such as S-modules [3] or symmetric and diagram spectra

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with the positive stable model structure [9,14], are indispensable in brave new algebraic geometry [18, §2.4]. Lewis–Mandell [6] and more recently the author [10–12] developed some techniques to deal with non-cofibrant tensor units under mild extra assumptions. One of them is the *very strong unit axiom*, which is the strengthening of the unit axiom where X can be any object. This new axiom holds in all monoidal model categories known to the author since, in all of them, tensoring with a cofibrant object preserves weak equivalences, see Corollary 9 below.

In this paper, we prove that we can equip any suitable monoidal model category with a different model structure with the same weak equivalences where the tensor unit is cofibrant. This new model structure is minimal in a certain sense.

Any monoidal category has an underlying set functor $\mathscr{M}(\mathbf{1},-)\colon \mathscr{M} \to \mathrm{Set}$. A map in $\mathscr{M}$ is said to be *surjective* if the induced map on underlying sets is surjective. Notice that the tensor unit is cofibrant in $\mathscr{M}$ if and only if all trivial fibrations are surjective.

Theorem 1. *Any combinatorial monoidal model category $\mathscr{M}$ satisfying the very strong unit axiom admits a combinatorial monoidal model structure $\tilde{\mathscr{M}}$ with the same weak equivalences and whose trivial fibrations are the surjective trivial fibrations in $\mathscr{M}$. If $\mathscr{M}$ is right or left proper then so is $\tilde{\mathscr{M}}$. If $\mathscr{M}$ is symmetric and satisfies the monoid axiom then so does $\tilde{\mathscr{M}}$.*

Example 2. Let $\mathscr{M} = \mathrm{Sp}^\Sigma$ be the category of symmetric spectra of simplicial sets equipped with the positive stable model structure [14, Proposition 3.1], where the sphere spectrum $\mathbf{1} = S$ is not cofibrant. It is proper, symmetric, and satisfies the monoid axiom. The very strong unit axiom is a consequence of Corollary 9, [5, Lemma 5.4.4], and the fact that cofibrations in the positive stable model structure are also cofibrations in the ordinary stable model structure. Theorem 1 applies and trivial fibrations in $\tilde{\mathscr{M}}$ are the maps $f\colon X \to Y$ such that $f_n\colon X_n \to Y_n$ is a trivial Kan fibration for any $n > 0$ and $f_0\colon X_0 \to Y_0$ is surjective on vertices.

The model structure $\tilde{\mathscr{M}}$ is, strictly, between the ordinary and the positive stable model structures. Indeed, $\tilde{\mathscr{M}} \neq \mathscr{M}$ since $\mathbf{1}$ is cofibrant in the former but not in the latter. We now exhibit a trivial fibration in $\tilde{\mathscr{M}}$ which is not an ordinary stable trivial fibration. Let X be a fibrant replacement of the sphere spectrum in the ordinary stable model structure and let $X' \subset X$ be the subspectrum with $X'_n = X_n$ for $n > 0$ and $X'_0 =$ the discrete simplicial set with the same vertices as X_0. The Kan complex X_0 is not discrete since its homotopy groups are the stable homotopy groups of the sphere spectrum, therefore $X'_0 \subset X_0$ is not a trivial Kan fibration. In particular, $X' \subset X$ is a trivial fibration in $\tilde{\mathscr{M}}$ which is not an ordinary stable trivial fibration.

As far as we know, the model structure $\tilde{\mathscr{M}}$ on symmetric spectra is new. Despite being so close to the positive stable model structure, $\tilde{\mathscr{M}}$ is not suitable to deal with commutative ring spectra precisely because the sphere spectrum is cofibrant. This observation also applies in the examples below.

We will actually prove the following result, with weaker but uglier hypotheses. Denote by $\varnothing$ the initial object of $\mathscr{M}$.

Theorem 3. *Let $\mathscr{M}$ be a cofibrantly generated monoidal model category satisfying the very strong unit axiom for a certain cofibrant resolution $q\colon \tilde{\mathbf{1}} \overset{\sim}{\to} \mathbf{1}$. Let*

$$\tilde{\mathbf{1}} \amalg \mathbf{1} \overset{j}{\rightarrowtail} C \underset{\sim}{\overset{p}{\to}} \mathbf{1}$$

be a factorization of $(q, \mathrm{id}_{\mathbf{1}})\colon \tilde{\mathbf{1}} \amalg \mathbf{1} \to \mathbf{1}$ into a cofibration followed by a weak equivalence in $\mathscr{M}$ and let $i_1\colon \tilde{\mathbf{1}} \to \tilde{\mathbf{1}} \amalg \mathbf{1}$ be the inclusion of the first factor of the coproduct. Assume that $\mathscr{M}$ has sets I and J of generating cofibrations and generating trivial cofibrations, respectively, such that the domains of I are small relative to $\tilde{I}$-cell for $\tilde{I} = I \cup \{\varnothing \to \mathbf{1}\}$ and $\tilde{\mathbf{1}}$ and the domains of J are small relative to $\tilde{J}$-cell for