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# Nonseparable growth of the integers supporting a measure

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Assuming $\mathfrak{b} = \mathfrak{c}$ (or some weaker statement), we construct a compactification $\gamma\omega$ of $\omega$ such that its remainder $\gamma\omega \setminus \omega$ is nonseparable and carries a strictly positive measure.



## 1. Introduction

We consider here compactifications $\gamma\omega$ of the discrete space $\omega$. Given a compact space $K$, we say that $K$ is a growth of $\omega$ if there is a compactification $\gamma\omega$ with the remainder $\gamma\omega \setminus \omega$ homeomorphic to the space $K$. It is not difficult to check that every separable compactum is a growth of $\omega$.

By a well-known theorem due to Parovičenko [10], under the continuum hypothesis every compact space $K$ of topological weight $\leq \mathfrak{c}$ is a continuous image of the remainder $\beta\omega\setminus\omega$ of the Čech–Stone compactification of $\omega$ and, consequently, $K$ is homeomorphic to the remainder $\gamma\omega \setminus \omega$ of some compactification $\gamma\omega$.

Let $S$ be the Stone space of the measure algebra which may be seen as the quotient of $Bor(2^\omega)$ modulo the ideal of null sets. Then $S$ is a nonseparable compact space that carries a strictly positive (regular probability Borel) measure, i.e. a measure that is positive on every nonempty open subset of $S$. Since the topological weight of $S$ is $\mathfrak{c}$, CH implies that $S$ is a growth of $\omega$. In fact, this may be done in such a way that the

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canonical measure on $S$ is defined by the asymptotic density defined for subsets of $\omega$, see Frankiewicz and Gutek [5].

Dow and Hart [3] proved that the space $S$ is not a growth of $\omega$ if one assumes the Open Coloring Axiom (OCA). Therefore it seems to be interesting to investigate whether one can always, in the usual set theory, construct a compactification $\gamma\omega$ such that the remainder $\gamma\omega \setminus \omega$ is nonseparable but carries a strictly positive measure.

If a compact space carries a strictly positive measure then it satisfies *ccc*; the converse does not hold as was already demonstrated by Gaifman [6]. Later Bell [2], van Mill [9], Todorčević [11] constructed several interesting examples of compactifications of $\omega$ having nonseparable *ccc* remainders; cf. Todorčević [12]. It seems that the structure of all those examples exclude the possibility that *ccc* could be strengthened to saying that the remainder in question supports a measure, see e.g. Lemma 3.2 in Džamonja and Plebanek [4].

Before we state our main result we need to fix some notation and terminology concerning the set-theoretic assumption that we use. We denote by $\lambda$ the usual product measure on the Cantor set $2^\omega$. Let $\mathcal{E}$ be the ideal of subsets $A$ of $2^\omega$ for which $\lambda(\overline{A}) = 0$. Recall that the covering number $\mathrm{cov}(\mathcal{E})$ is the least cardinality of a covering of $2^\omega$ by sets from $\mathcal{E}$; cf. Bartoszyński and Shelah [1] for cardinal invariants of the ideal $\mathcal{E}$.

Throughout this paper we write $\kappa_0 = \mathrm{cov}(\mathcal{E})$ and $\kappa = \mathrm{cof}[\kappa_0]^{\le\omega}$, where $[\kappa_0]^{\le\omega}$ is the partially ordered set of all countable subsets of $\kappa_0$. In other words, $\kappa$ is the least size of a family $\mathcal{J} \subseteq [\kappa_0]^{\le\omega}$ such that every countable subset of $\kappa_0$ is contained in some $J \in \mathcal{J}$.

Our set-theoretic assumption $(*)$ involves also $\mathfrak{b}$, the familiar bounding number and reads as follows:

$$\kappa = \mathrm{cof}[\kappa_0]^{\le\omega} \le \mathfrak{b}. \tag{$*$}$$

**Theorem 1.1.** *Assuming $(*)$ there is a compactification $\gamma\omega$ of the set of natural numbers such that its remainder $\gamma\omega \setminus \omega$ is not separable but carries a strictly positive regular probability Borel measure.*

Note that $(*)$ holds whenever $\mathfrak{b} = \mathfrak{c}$ or $\kappa_0 = \omega_1$. We do not know whether Theorem 1.1 can be proved in the usual set theory. In connection with the result of Dow and Hart mentioned above it is worth remarking that OCA implies $\mathfrak{b} = \omega_2$ and we can further assume $\omega_2 = \mathfrak{c}$ (see Moore [8]) so the compactification we construct here may exist even when the Stone space of the measure algebra is not a growth of $\omega$.

We remark that in our proof of Theorem 1.1 we construct $\gamma\omega$ such that $\gamma\omega \setminus \omega$ supports a measure $\mu$ of countable Maharam type (meaning that $L_1(\mu)$ is separable). We might, however, modify the construction so that the resulting $\mu$ will be of type $\kappa$.

The paper is organized as follows. In Section 2 we formulate Theorem 1.1 in terms of subalgebras of $P(\omega)$ and finitely additive measures defined on them, see Theorem 2.2. Section 3 describes a recursive construction using $(*)$ that leads to Theorem 2.2. The key ingredient of the argument is stated as Lemma 3.2 — its proof is postponed to the final Section 4.

## 2. Compactifications and Boolean algebras

We denote by *fin* the family of finite subsets of $\omega$. In the sequel, we shall consider Boolean algebras (of sets) $\mathfrak{A}$ such that $\mathit{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$. Every such an algebra $\mathfrak{A}$ determines a compactification $K_\mathfrak{A}$ of $\omega$, where $K_\mathfrak{A}$ may be seen as the Stone space of all ultrafilters on $\mathfrak{A}$. Then the algebra $\mathrm{Clop}(K^*_\mathfrak{A})$ of the clopen subsets of the remainder $K^*_\mathfrak{A} = K_\mathfrak{A} \setminus \omega$ is isomorphic to the quotient algebra $\mathfrak{A}/\mathit{fin}$. Hence $K^*_\mathfrak{A}$ is not separable if and only if $\mathfrak{A}/\mathit{fin}$ is not $\sigma$-centered.

Given an algebra $\mathfrak{A}$ such that $\mathit{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$, we shall consider finitely additive probability measures $\mu$ on $\mathfrak{A}$ that vanish on finite sets. Such a measure $\mu$ defines, via the Stone isomorphism, a finitely additive measure $\widehat{\mu}$ on $\mathrm{Clop}(K_\mathfrak{A})$. Then $\widehat{\mu}$ extends uniquely to a regular probability Borel measure $\overline{\mu}$ on $K_\mathfrak{A}$ such that $\overline{\mu}(K^*_\mathfrak{A}) = 1$. Note that the resulting Borel measure will be strictly positive whenever $\mu$ has the property mentioned in the following definition.