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Discrete subsets and convergent sequences in topological groups

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We show that (1) assuming there is no rapid ultrafilter, every countable nondiscrete maximally almost periodic topological group contains a discrete nonclosed subset which is a convergent sequence in some weaker totally bounded group topology, (2) every infinite Abelian totally bounded topological group contains such a discrete subset (in ZFC), and (3) if an extremally disconnected topological group contains such a discrete subset, then there is a selective ultrafilter.

1. Introduction

In [8], O. Sipacheva obtained partial solutions to Arhangel'skiĭ's question of whether there exists in ZFC a countable nondiscrete extremally disconnected topological group [1] and to Protasov's question of whether there exists in ZFC a countable nondiscrete topological group in which every discrete subset is closed [6]. It had been known previously that the existence of an extremally disconnected topological group containing a countable discrete nonclosed subset implies the existence of a *P*-point, and similar result holds for a nondiscrete group topology $\mathcal{T}$ on $\bigoplus_\omega \mathbb{Z}_2$ which is finer than the direct sum topology, has no discrete subset

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with a unique accumulation point, and satisfies one more condition [10]. (The *direct sum* topology is the one induced by the product topology.) Sipacheva showed that the existence of a nondiscrete extremally disconnected group topology on $\bigoplus_\omega \mathbb{Z}_2$ finer than the direct sum topology implies the existence of a rapid ultrafilter, and similar result holds for a group topology on $\bigoplus_\omega \mathbb{Z}_2$ which is finer than the direct sum topology and has no discrete subset with a unique accumulation point.

In this paper we show that

(1) assuming there is no rapid ultrafilter, every countable nondiscrete maximally almost periodic topological group contains a discrete nonclosed subset which is a convergent sequence in some weaker totally bounded group topology,

(2) every infinite Abelian totally bounded topological group contains such a discrete subset (in ZFC), and

(3) the existence of an extremally disconnected topological group containing such a discrete subset implies the existence of a selective ultrafilter.

Notice that such a subset has a unique accumulation point and converges to that point in that weaker topology.

Recall that a nonprincipal ultrafilter p on ω is

(i) a *P-point* if for every partition $\{A_n : n < \omega\}$ of ω with $A_n \notin p$, there is $A \in p$ such that $|A \cap A_n| < \omega$ for all n,

(ii) *selective* if for every partition $\{A_n : n < \omega\}$ of ω with $A_n \notin p$, there is $A \in p$ such that $|A \cap A_n| \le 1$ for all n, and

(iii) *rapid* if for every partition $\{A_n : n < \omega\}$ of ω with finite A_n, there is $A \in p$ such that $|A \cap A_n| \le n$ for all n.

Clearly, every selective ultrafilter is rapid. MA implies the existence of P-points and rapid ultrafilters. However, it is consistent with ZFC that there is no P-point [7, VI, §4], and it is consistent with ZFC that there is no rapid ultrafilter [5].

A topological group is *totally bounded* (*maximally almost periodic*) if it can be topologically and algebraically (continuously and algebraically) embedded into a compact group. All topologies are assumed to be Hausdorff.

2. Rapid ultrafilters

Every group topology on a countable group can be weakened to a metrizable (= first countable) group topology [2]. Consequently, every maximally almost periodic group topology on a countable group can be weakened to a metrizable totally bounded group topology.

In this section we prove the following result.

Theorem 2.1. *Assume there is no rapid ultrafilter. Let $(G, \mathcal{T})$ be a countable nondiscrete maximally almost periodic topological group and let $\mathcal{T}_0$ be a metrizable totally bounded group topology on G weaker than $\mathcal{T}$. Then $(G, \mathcal{T})$ contains a discrete nonclosed subset which is a convergent sequence in $\mathcal{T}_0$.*

As a consequence we obtain that

Corollary 2.2. *It is consistent with ZFC that every countable nondiscrete maximally almost periodic topological group contains a discrete nonclosed subset which is a convergent sequence in some weaker totally bounded group topology.*

The proof of Theorem 2.1 involves the following result [11, Theorem 3.1].

Theorem 2.3. *Let G be a countably infinite metrizable totally bounded topological group. Then there exist a sequence $(m_n)_{n<\omega}$ of integers ≥ 2 and a homeomorphism $h : G \to \bigoplus_{n<\omega} \mathbb{Z}_{m_n}$ with $h(1) = 0$ such that*