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A note on semitopological groups and paratopological groups ☆

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ABSTRACT

We give a description of the T_2-reflection of a semitopological group G as the quotient of G with respect to a certain subgroup of G. This answers an open problem of M. Tkachenko (Problem 4.1 which appears in [6]). We also show that if G is an ω-cellular regular ω-balanced paratopological group then the following conditions are equivalent:

(1) $Sm(G) \leq \omega$;
(2) $Hs(G) \leq \omega$;
(3) $Ir(G) \leq \omega$.

1. Introduction

Recall that a *topological group* G is a group with a (Hausdorff) topology such that the product mapping of $G \times G$ into G is jointly continuous and the inverse mapping of G onto itself associating x^{-1} with $x \in G$ is continuous. A *paratopological group* is a group with a topology such that the multiplication is jointly continuous. However, there exists a paratopological group which is not a topological group; Sorgenfrey line [3, Example 1.2.2] is such an example. *Semitopological groups* are groups with a topology in which the left and right translations are continuous. The following definition appears in [6]. A class $\mathcal{C}$ of spaces is a *PS-class* if it contains arbitrary products of its elements, is hereditary with respect to taking subspaces, and contains a one-point space. Let $\mathcal{C}$ be a PS-class of spaces and let $\varphi_G^{\mathcal{C}} : G \to H$ be a continuous surjective homomorphism of semitopological groups. The pair $(H, \varphi_G^{\mathcal{C}})$ is called a *$\mathcal{C}$-reflection* of G if $H \in \mathcal{C}$ and for every continuous mapping $f : G \to X$ to a space $X \in \mathcal{C}$, there exists a continuous mapping $h : H \to X$ such that $f = h \circ \varphi_G^{\mathcal{C}}$ [6].

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Abusing terminology we will usually refer to $H = \varphi_G^{\mathcal{C}}(G)$ as a $\mathcal{C}$-reflection of G. For a semitopological group G, M. Tkachenko [6] stated the existence of T_i-reflection $T_i(G)$ for $i \in \{0,1,2,3\}$, as well as the regular reflection $Reg(G)$ and Tychonoff reflections $Tych(G)$, respectively. In [6], Tkachenko also gave a description of the T_i-reflection $T_i(G)$, for $i = 0, 1$, of a semitopological group G as the quotient of G with respect to a certain subgroup of G defined in internal terms. The following problem appears in [6, Problem 4.1].

Problem. ([6, Problem 4.1]) Describe in internal terms the kernel of the canonical homomorphism $\varphi_{G,2}$ of a semitopological group G onto $T_2(G)$.

The above problem is answered in this note.

Recall that a space X is called *ω-cellular* [1] if every family γ consisting of G_δ-sets in X contains a subfamily γ_0 such that $\overline{\bigcup \gamma_0} = \overline{\bigcup \gamma}$ and $|\gamma_0| \le \omega$. Recall that the *Hausdorff number* of a Hausdorff semitopological group G with the identity e, denoted by $Hs(G)$, is the minimum cardinal number κ such that for every neighbourhood U of e in G, there exists a family γ of neighbourhoods of e such that $\bigcap\{VV^{-1} : V \in \gamma\} \subset U$ and $|\gamma| \le \kappa$ [7]. The *symmetry number* [5] of a T_1 semitopological group G with the identity e, denoted by $Sm(G)$, is the minimum cardinal number κ such that for every neighbourhood U of e in G, there exists a family γ of neighbourhoods of e such that $\bigcap\{V^{-1} : V \in \gamma\} \subset U$ and $|\gamma| \le \kappa$. The *index of regularity* of a regular semitopological group G with the neutral element e, denoted by $Ir(G)$, is the minimum cardinal number κ such that for every neighbourhood U of e in G, one can find a neighbourhood V of e and a family γ of neighbourhoods of e in G such that $\bigcap\{VW^{-1} : W \in \gamma\} \subset U$ and $|\gamma| \le \kappa$ [7]. If G is a regular paratopological group, then it is obvious that $Sm(G) \le Hs(G) \le Ir(G)$. In [5], there is an open problem that find conditions under which a Hausdorff paratopological group G satisfies $Sm(G) = Hs(G)$. In this note, we show that if G is an ω-cellular regular ω-balanced paratopological group then the following conditions are equivalent:

(1) $Sm(G) \le \omega$;
(2) $Hs(G) \le \omega$;
(3) $Ir(G) \le \omega$.

To get the above conclusion, we need the following result (Theorem 9). Let G be an ω-cellular regular paratopological group with the neutral element e and let $\mathcal{V}$ be a countable family of open neighbourhoods of e in G. If G is ω-balanced and $Sm(G) \le \omega$, then there exists an open continuous homomorphism $p : G \to K$ of G onto a first-countable regular paratopological group H such that $ker(p) \subset \bigcap \mathcal{V}$, $\overline{U} = p^{-1}(\overline{p(U)})$ for each $U \in \mathcal{V}$, and for each $U \in \mathcal{V}$ there is an open neighbourhood W_U of the neutral element e_K in K such that $p^{-1}(W_U) \subset U$.

The set of all positive integers is denoted by $\mathbb{N}$ and ω is $\mathbb{N} \cup \{0\}$. In notation and terminology we will follow [1] and [3]. Every regular topological space is a T_1-space in this article.

2. Main results

In order to describe in internal terms the kernel of the canonical homomorphism $\varphi_{G,2}$ of a semitopological group G onto the T_2-reflection $T_2(G)$ of G, we need the following lemmas.

Lemma 1. *Let G, K be semitopological groups. If $f : G \to K$ is a continuous homomorphism such that $|f(G)| > 1$ and K is a Hausdorff semitopological group, then there is an open neighbourhood V of the neutral element e_G in G such that $VV^{-1} \neq G$.*

Proof. Let $p, q \in f(G)$ such that $p \neq q$. Since K is a Hausdorff space, there are open sets O_p and O_q of K such that $O_p \cap O_q = \emptyset$ and $p \in O_p$, $q \in O_q$. Let $x, y \in G$ such that $f(x) = p$ and $f(y) = q$. Thus $x \neq y$.