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Products of bounded subsets of paratopological groups

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ABSTRACT

We prove that if B_i is a bounded subset of a totally ω-narrow paratopological group G_i, where $i \in I$, then $\prod_{i\in I} B_i$ is bounded in $\prod_{i\in I} G_i$. The same conclusion remains valid in the case of products of bounded subsets of Hausdorff commutative paratopological groups with countable Hausdorff number or products of Lindelöf paratopological groups. In fact, we show that if B is a bounded subset of a paratopological group G satisfying one of the conditions (a)–(d) below, then B is strongly bounded in G:

(a) G is totally ω-narrow;
(b) G is commutative, Hausdorff, and has countable Hausdorff number;
(c) G is saturated and weakly Lindelöf;
(d) G is Lindelöf.

These results imply that boundedness of subsets is productive in the classes of paratopological groups listed in (a)–(d).

1. Introduction

The product of an arbitrary family of pseudocompact topological groups is pseudocompact—this is the celebrated theorem proved by Comfort and Ross in [7]. Recently, A. Ravsky [13] proved a similar result for products of pseudocompact paratopological groups. To be more accurate, we have to reformulate Ravsky's theorem as follows: *An arbitrary product of feebly compact paratopological groups is feebly compact.* As usual, a space X is called *feebly compact* if every locally finite family of open sets in X is finite. Feebly compact spaces are not assumed to satisfy any separation axiom, while pseudocompact spaces are necessarily

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Tychonoff. It is clear that feeble compactness and pseudocompactness coincide in Tychonoff spaces, so the former notion is a ‘right’ extension of the latter one to non-Tychonoff spaces.

The Comfort–Ross theorem on the productivity of pseudocompactness in topological groups admits several extensions to wider classes of spaces (see [1,6,31,32]). Another generalization was obtained by the second-listed author in [22, Theorem 2.2]: If B_i is a bounded subset of a topological group G_i, for each $i \in I$, then $\prod_{i\in I} B_i$ is bounded in $\prod_{i\in I} G_i$.

Let us recall that a subset B of a space X is said to be *bounded* in X if every locally finite family of open sets in X contains only finitely many elements that meet B. Hence boundedness is a *relative* version of feeble compactness. It is clear from the definition that a subset B of a *Tychonoff* space X is bounded in X if and only if every continuous real-valued function defined on X is bounded on B.

With Ravsky’s theorem in mind, it is natural to ask whether boundedness remains productive for subsets of paratopological groups (see [19, Problem 2.16] and [26, Problem 7.1]). We prove in Theorem 2.8 that this is indeed the case if the paratopological groups are additionally assumed to be *totally ω-narrow* (the factors in the theorem are not assumed to satisfy any separation axiom). The same conclusion is valid for products of bounded subsets of Hausdorff commutative paratopological groups with countable Hausdorff number (Corollary 3.10) and for products of bounded subsets of saturated, weakly Lindelöf paratopological groups (Corollary 3.14). The latter fact implies that a similar conclusion holds for products of bounded subsets of precompact paratopological groups (see Corollary 3.16).

The key technical notion in this article is the one called *strong boundedness* (see Definition 2.3). It is known that every bounded subset of a topological group is strongly bounded in the group [24, Theorem 2]. We show that in all the aforementioned cases, a bounded subset of a paratopological group turns out to be strongly bounded in the group. Then we apply the fact that strong boundedness is productive for subsets of topological spaces [22, Theorem 2.6]. The question of whether a bounded subset B of an arbitrary paratopological group G is strongly bounded in G remains open, even if B is countably compact (see Problem 5.1).

Section 5 of the article contains several open problems on bounded sets in paratopological groups which are supplied with brief comments.

1.1. Notation, terminology, and preliminary facts

A *paratopological* group is a group with a topology such that multiplication on the group is jointly continuous. The wording *an isomorphism of paratopological groups* does not necessarily mean that the isomorphism in question is continuous.

If τ is the topology of a paratopological group G, then the family

$$\tau^{-1} = \{U^{-1} : U \in \tau\}$$

is also a topology on G and $G' = (G, \tau^{-1})$ is again a paratopological group *conjugated* to G. It is clear that the inversion on G is a homeomorphism of G onto G'. The upper bound $\tau^* = \tau \vee \tau^{-1}$ is a topological group topology on G and $G^* = (G, \tau^*)$ is a topological group *associated* to G.

A paratopological (topological) group G is said to be *ω-narrow* if for every neighborhood U of the neutral element in G, there exists a countable set $C \subseteq G$ such that $CU = G = UC$. We call a paratopological group G *totally ω-narrow* if the topological group G^* associated to G is ω-narrow. In topological groups, total ω-narrowness and ω-narrowness coincide, but the Sorgenfrey line is an example of an ω-narrow (even Lindelöf) paratopological group which fails to be totally ω-narrow.

A paratopological group G is *ω-balanced* if for every neighborhood U of the identity e in G, one can find a countable family γ of open neighborhoods of e in G such that for every $x \in G$, there exists $V \in \gamma$ with $xVx^{-1} \subseteq U$. It is clear that every paratopological Abelian group is ω-balanced.