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On topological spaces and topological groups with certain local countable networks

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ABSTRACT

Being motivated by the study of the space $C_c(X)$ of all continuous real-valued functions on a Tychonoff space X with the compact–open topology, we introduced in [16] the concepts of a *cp*-network and a *cn*-network (at a point x) in X. In the present paper we describe the topology of X admitting a countable *cp*- or *cn*-network at a point $x \in X$. This description applies to provide new results about the strong Pytkeev property, already well recognized and applicable concept originally introduced by Tsaban and Zdomskyy [44]. We show that a Baire topological group G is metrizable if and only if G has the strong Pytkeev property. We prove also that a topological group G has a countable *cp*-network if and only if G is separable and has a countable *cp*-network at the unit. As an application we show, among the others, that the space $D'(\Omega)$ of distributions over open $\Omega \subseteq \mathbb{R}^n$ has a countable *cp*-network, which essentially improves the well known fact stating that $D'(\Omega)$ has countable tightness. We show that, if X is an $\mathcal{MK}_\omega$-space, then the free topological group $F(X)$ and the free locally convex space $L(X)$ have a countable *cp*-network. We prove that a topological vector space E is p-normed (for some $0 < p \leq 1$) if and only if E is Fréchet–Urysohn and admits a fundamental sequence of bounded sets.

1. Introduction

All topological spaces are assumed to be Hausdorff. Various topological properties generalizing metrizability have been studied intensively by topologists and analysts, especially like first countability, Fréchet–Urysohness, sequentiality and countable tightness (see [9,25]). It is well-known that

$$\text{metric} \Longrightarrow \begin{matrix}\text{first}\\ \text{countable}\end{matrix} \Longrightarrow \begin{matrix}\text{Fréchet–}\\ \text{Urysohn}\end{matrix} \Longrightarrow \text{sequential} \Longrightarrow \begin{matrix}\text{countable}\\ \text{tight}\end{matrix},$$

and none of these implications can be reversed.

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One of the most immediate extensions of the class of separable metrizable spaces are the classes of cosmic and $\aleph_0$-spaces in sense of Michael [30].

Definition 1.1. ([30]) A topological space X is called

- *cosmic*, if X is a regular space with a countable network (a family $\mathcal{N}$ of subsets of X is called a *network* in X if, whenever $x \in U$ with U open in X, then $x \in N \subseteq U$ for some $N \in \mathcal{N}$);
- an $\aleph_0$-*space*, if X is a regular space with a countable k-network (a family $\mathcal{N}$ of subsets of X is called a k-*network* in X if, whenever $K \subseteq U$ with K compact and U open in X, then $K \subseteq \bigcup \mathcal{F} \subseteq U$ for some finite family $\mathcal{F} \subseteq \mathcal{N}$).

These classes of topological spaces were intensively studied in [20,23,30] and [31].

Having in mind the Nagata–Smirnov metrization theorem, Okuyama [38] and O'Meara [34] introduced the classes of σ-spaces and $\aleph$-spaces, respectively.

Definition 1.2. A topological space X is called

- ([38]) a σ-*space* if X is regular and has a σ-locally finite network;
- ([34]) an $\aleph$-*space* if X is regular and has a σ-locally finite k-network.

Any metrizable space X is an $\aleph$-space. O'Meara [33] proved that an $\aleph$-space which is either first countable or locally compact is metrizable. Every compact subset of a σ-space is metrizable [32]. Further results see [22].

Pytkeev [40] proved that every sequential space satisfies the property, known actually as the *Pytkeev property*, which is stronger than countable tightness: a topological space X has the *Pytkeev property* if for each $A \subseteq X$ and each $x \in \overline{A} \setminus A$, there are infinite subsets $A_1, A_2, \dots$ of A such that each neighborhood of x contains some A_n. Tsaban and Zdomskyy [44] strengthened this property as follows. A topological space X has the *strong Pytkeev property* if for each $x \in X$, there exists a countable family $\mathcal{D}$ of subsets of X, such that for each neighborhood U of x and each $A \subseteq X$ with $x \in \overline{A} \setminus A$, there is $D \in \mathcal{D}$ such that $D \subseteq U$ and $D \cap A$ is infinite. Next, Banakh [1] introduced the concept of the Pytkeev network in X as follows: A family $\mathcal{N}$ of subsets of a topological space X is called a *Pytkeev network at a point* $x \in X$ if $\mathcal{N}$ is a network at x and for every open set $U \subseteq X$ and a set A accumulating at x there is a set $N \in \mathcal{N}$ such that $N \subseteq U$ and $N \cap A$ is infinite. Hence X has the strong Pytkeev property if and only if X has a countable Pytkeev network at each point $x \in X$.

In [18] we proved that the space $C_c(X)$ has the strong Pytkeev property for every Čech-complete Lindelöf space X. For the proof of this result we constructed a family $\mathcal{D}$ of sets in $C_c(X)$ such that for every neighborhood $U_\mathbf{0}$ of the zero function $\mathbf{0}$ the union $\bigcup\{D \in \mathcal{D} : \mathbf{0} \in D \subseteq U_\mathbf{0}\}$ is a neighborhood of $\mathbf{0}$ (see the condition (**D**) in [18]). Having in mind this idea for $C_c(X)$ we proposed in [16] the following types of networks which will be applied in the sequel.

Definition 1.3. ([16]) A family $\mathcal{N}$ of subsets of a topological space X is called

- a *cn-network at a point* $x \in X$ if for each neighborhood O_x of x the set $\bigcup\{N \in \mathcal{N} : x \in N \subseteq O_x\}$ is a neighborhood of x; $\mathcal{N}$ is a *cn-network* in X if $\mathcal{N}$ is a *cn*-network at each point $x \in X$;
- a *ck-network at a point* $x \in X$ if for any neighborhood O_x of x there is a neighborhood U_x of x such that for each compact subset $K \subseteq U_x$ there exists a finite subfamily $\mathcal{F} \subseteq \mathcal{N}$ satisfying $x \in \bigcap \mathcal{F}$ and $K \subseteq \bigcup \mathcal{F} \subseteq O_x$; $\mathcal{N}$ is a *ck-network* in X if $\mathcal{N}$ is a *ck*-network at each point $x \in X$;