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A note on stable complex structures on real vector bundles over manifolds

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ABSTRACT

Let M be an n-dimensional closed oriented smooth manifold with $n \equiv 0 \bmod 8$, ξ be a real vector bundle over M. Suppose that ξ admits a stable complex structure over the $(n-1)$-skeleton of M. Then the necessary and sufficient conditions for ξ to admit a stable complex structure over M are given in terms of the characteristic classes of ξ and M. As an application, we obtain the criteria to determine which real vector bundle over 8-dimensional manifold admits a stable complex structure.

1. Introduction

First we introduce some notations. For a topological space X, let $Vect_{\mathbb{C}}(X)$ (resp. $Vect_{\mathbb{R}}(X)$) be the set of isomorphic classes of complex (resp. real) vector bundles on X, and let $\widetilde{K}(X)$ (resp. $\widetilde{KO}(X)$) be the reduced KU-group (resp. reduced KO-group) of X, which is the set of stable equivalent classes of complex (resp. real) vector bundles over X. For a map $f: X \to Y$ between topological spaces X and Y, denote by

$$f_u^*: \widetilde{K}(Y) \to \widetilde{K}(X) \quad \text{and} \quad f_o^*: \widetilde{KO}(Y) \to \widetilde{KO}(X)$$

the induced homomorphisms. For $\xi \in Vect_{\mathbb{R}}(X)$ (resp. $\eta \in Vect_{\mathbb{C}}(X)$), we will denote by $\tilde{\xi} \in \widetilde{KO}(X)$ (resp. $\tilde{\eta} \in \widetilde{K}(X)$) the stable class of ξ (resp. η) (cf. Hilton [8, p. 62]), $w_i(\xi)$ (resp. $p_i(\xi)$) the i-th Stiefel–Whitney class (resp. Pontrjagin class) of ξ, $ch(\tilde{\eta})$ the Chern character of $\tilde{\eta}$. In particular, if X is a smooth manifold,

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then $w_i(X) = w_i(TX)$ (resp. $p_i(X) = p_i(TX)$) is the i-th Stiefel–Whitney class (resp. Pontrjagin class) of X, where TX is the tangent bundle of X.

It is known that there are natural homomorphisms

$$\begin{aligned} \tilde{c}_X : \widetilde{KO}(X) \to \widetilde{K}(X) & \qquad \text{the complexification,} \\ \tilde{r}_X : \widetilde{K}(X) \to \widetilde{KO}(X) & \qquad \text{the real reduction,} \\ - : \widetilde{K}(X) \to \widetilde{K}(X) & \qquad \text{conjugation,} \end{aligned}$$

which induced from

$$\begin{aligned} c_X : Vect_{\mathbb{R}}(X) \to Vect_{\mathbb{C}}(X) & \qquad \text{the complexification,} \\ r_X : Vect_{\mathbb{C}}(X) \to Vect_{\mathbb{R}}(X) & \qquad \text{the real reduction,} \\ - : Vect_{\mathbb{C}}(X) \to Vect_{\mathbb{C}}(X) & \qquad \text{conjugation,} \end{aligned}$$

respectively, and they satisfy the following relations:

$$\tilde{r}_X \circ \tilde{c}_X(\tilde{\xi}) = 2\tilde{\xi}, \qquad \text{for any } \tilde{\xi} \in \widetilde{KO}(X); \tag{1.1}$$

$$\tilde{c}_X \circ \tilde{r}_X(\tilde{\eta}) = \tilde{\eta} + \bar{\tilde{\eta}}, \qquad \text{for any } \tilde{\eta} \in \widetilde{K}(X). \tag{1.2}$$

Let $\xi \in Vect_{\mathbb{R}}(X)$ be a real vector bundle over X. We say that ξ admits a *stable complex structure* over X if there exists a complex vector bundle η over X such that $\tilde{r}_X(\tilde{\eta}) = \tilde{\xi}$, that is $\tilde{\xi} \in \operatorname{Im} \tilde{r}_X$.

Let $U = \lim_{n\to\infty} U(n)$ (resp. $SO = \lim_{n\to\infty} SO(n)$) be the stable unitary (resp. special orthogonal) group. Denote by $\Gamma = SO/U$. Let X^q be the q-skeleton of X, and denote by $i\colon X^q \to X$ the inclusion map of q-skeleton of X. Suppose that ξ admits a stable complex structure over X^q, that is there exists a complex vector bundle η over X^q such that

$$i_o^*(\tilde{\xi}) = \tilde{r}_{X^q}(\tilde{\eta}).$$

Then the obstruction to extending η over the $(q+1)$-skeleton of X is denoted by

$$\mathfrak{o}_{q+1}(\eta) \ \in \ H^{q+1}(X, \pi_q(\Gamma))$$

where

$$\pi_q(\Gamma) = \begin{cases} \mathbb{Z}, & q \equiv 2 \bmod 4, \\ \mathbb{Z}/2, & q \equiv 0, \ -1 \bmod 8, \\ 0, & \text{otherwise.} \end{cases}$$

(cf. Bott [2] or Massey [9, p. 560]).

If $q \equiv 2 \bmod 4$, that is the coefficient group $\pi_q(\Gamma) = \mathbb{Z}$, the obstructions $\mathfrak{o}_{q+1}(\eta)$ have been investigated by Massey [9, Theorem I]. Moreover, when the coefficient group $\pi_q(\Gamma) = \mathbb{Z}/2$, Massey [9, Theorem III] and Thomas [13, Theorem 1.2] determined the obstruction $\mathfrak{o}_8(\eta)$ in terms of characteristic classes and a secondary cohomology operation Ω (cf. E. Thomas [13]). The secondary cohomology operation Ω is difficult to calculate and it was calculated by the author only in special cases (cf. [13, Theorem 1.4]). Furthermore, M. Čadek, M. Crabb and J. Vanžura [4, Proposition 4.1(a)] expressed the obstruction $\mathfrak{o}_8(\eta)$ in terms of characteristic classes when X is an 8-dimensional *spin*^{c} manifold, Dessai [5, Theorem 1.9] does a similar work when X is a special 10-dimensional manifold.

In this paper, our main results are stated as follows.