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A new and simpler noncommutative central sets theorem

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Using dynamics, Furstenberg defined the concept of a central subset of positive integers and proved several powerful combinatorial properties of central sets. Later using the algebraic structure of the Stone–Čech compactification, Bergelson and Hindman, with the assistance of B. Weiss, generalized the notion of a central set to any semigroup and extended the most important combinatorial property of central sets to the central sets theorem. Currently the most powerful formulation of the central sets theorem is due to De, Hindman, and Strauss in [3, Corollary 3.10]. However their formulation of the central sets theorem for noncommutative semigroups is, compared to their formulation for commutative semigroups, complicated. In this paper I prove a simpler (but still equally strong) version of the noncommutative central sets theorem in Corollary 3.3.

1. Introduction

Furstenberg, in his book connecting dynamics to Ramsey theory, defined the concept of a *central subset of positive integers* [5, Definition 8.3] and proved several important properties of central sets using notions from topological dynamics. One such property is that whenever a central set is finitely partitioned, at least one cell of the partition contains a central set [5, Theorem 8.8]. Most of the remaining important properties of central sets are derivable from a powerful combinatorial theorem [5, Proposition 8.21]. (A bit later, in Theorem 1.3 on page 11, I state the most powerful formulation of [5, Proposition 8.21].) Furstenberg used his combinatorial theorem to prove Rado's theorem (see the sufficiency condition of [7, Theorem 5 on page 74] or Rado's published dissertation [13, Satz IV on page 445]) by showing that given any central set and any $m \times n$ matrix M, with integer entries, that satisfies the "columns condition" we can find a vector $\vec{x}$ all of whose entries are in the central set with $M\vec{x} = 0$.

Inspired by the fruitful interaction between ultrafilters on semigroups and Ramsey theory, Bergelson and Hindman, with the assistance of B. Weiss, later proved an algebraic characterization of a central subset of

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positive integers [1, Section 6]. This algebraic definition has several advantages over the original dynamical definition: one advantage is that the algebraic definition is simple and easily generalizes to any semigroup.

Definition 1.1. Let $(S, \cdot)$ be a discrete semigroup and let $A \subseteq S$. Then A is a ***central set*** if and only if there exists an idempotent p in the smallest ideal of βS with $A \in p$.

Another advantage of the algebraic definition is that central sets are "partition regular" — that is, in any finite partition of a central set at least one cell of the partition is a central set [5, Theorem 8.8] — is, because of standard properties of ultrafilters, immediate from the definition. More importantly, Furstenberg's combinatorial theorem [5, Proposition 8.21] follows from a relatively simple recursive construction.

Remark 1.2. The dynamical definition of a central set also extends naturally to an arbitrary semigroup; the fact that the algebraic and dynamical definitions are equivalent was proved by H. Shi and H. Yang in [14]. Besides this introduction I will usually not emphasize the dynamical point-of-view.

Using the algebraic structure of βS, the Stone–Čech compactification of a discrete semigroup S, and a more sophisticated recursive construction De, Hindman, and Strauss proved the (currently) strongest version of the central sets theorem in [3]. (The central sets theorem is what we shall call the main combinatorial property central sets satisfy.) We first state the central sets theorem for commutative semigroups: the statement of the noncommutative version is more complicated and forms the main focus of this paper.

In the statement of the commutative central sets theorem, and in the remainder of this paper, we let $\mathcal{P}_f(X)$ denote the collection of all nonempty finite subsets of a (typically nonempty) set X, and we let ${}^{A}B$ denote the collection of all functions with domain A and codomain B.

Theorem 1.3 *(Commutative central sets theorem). Let $(S, +)$ be a commutative semigroup and let $A \subseteq S$ be central. For typographical convenience we let $\mathcal{T} = {}^{\mathbb{N}}S$. Then there exist functions $\alpha\colon \mathcal{P}_f(\mathcal{T}) \to S$ and $H\colon \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ that satisfy the following two statements:*

(1) *If F and G are both in $\mathcal{P}_f(\mathcal{T})$ with $F \subsetneq G$, then $\max H(F) < \min H(G)$.*
(2) *Whenever m is a positive integer, $G_1, G_2, \ldots, G_m$ is a finite sequence in $\mathcal{P}_f(\mathcal{T})$ with $G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_m$ and for every $i \in \{1, 2, \ldots, m\}$ we have $f_i \in G_i$, then $\sum_{i=1}^{m} \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) \in A$.*

Proof. This was proved by De, Hindman, and Strauss as [3, Theorem 2.2]. □

In the same paper, De, Hindman, and Strauss also formulated and proved a strong version of the central sets theorem for arbitrary semigroups [3, Corollary 3.10]. The statement of this version of the central sets theorem is necessarily complicated because the underlying semigroup may be noncommutative. (When the underlying semigroup is commutative, [3, Corollary 3.10] reduces to Theorem 1.3.) In the case of noncommutativity it is usually *not sufficient* — for both combinatorial (see [2, Section 1]) and algebraic (see [12, Theorem 1.13]) reasons — to simply perform the obvious translation of Theorem 1.3 to an arbitrary semigroup. Roughly speaking, the proper translation requires splitting up each translate $\alpha(G_i)$ into several parts.

To better explain this difference in formulation let's first consider a special case of Theorem 1.3.

Corollary 1.4. *Let $(S, +)$ be a commutative semigroup and $A \subseteq S$ central. Then for every $F \in {}^{\mathbb{N}}S$ there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for every $f \in F$ we have $a + \sum_{t \in H} f(t) \in A$.*

Proof. Pick functions α and H as guaranteed by Theorem 1.3. Let $F \in {}^{\mathbb{N}}S$, put $m = 1$, and observe from conclusion (2) of Theorem 1.3 we have that for every $f \in F$