Topology and its Applications 189 (2015) 78–91

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

New lower bounds for the topological complexity of aspherical spaces ☆

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ARTICLE INFO

Article history:
Received 7 November 2014
Received in revised form 31 March 2015
Accepted 5 April 2015
Available online 13 April 2015

MSC:
primary 55M99, 55P20
secondary 55M30, 20J06, 68T40

Keywords:
Topological complexity
Aspherical spaces
Lusternik–Schnirelmann category
Cohomological dimension
Topological robotics
Infinite groups

ABSTRACT

We show that the topological complexity of an aspherical space X is bounded below by the cohomological dimension of the direct product $A \times B$, whenever A and B are subgroups of $\pi_1(X)$ whose conjugates intersect trivially. For instance, this assumption is satisfied whenever A and B are complementary subgroups of $\pi_1(X)$. This gives computable lower bounds for the topological complexity of many groups of interest (including semidirect products, pure braid groups, certain link groups, and Higman's acyclic four-generator group), which in some cases improve upon the standard lower bounds in terms of zero-divisors cup-length. Our results illustrate an intimate relationship between the topological complexity of an aspherical space and the subgroup structure of its fundamental group.

1. Introduction

Topological complexity is a numerical homotopy invariant introduced by Farber in the articles [13,14]. As well as being of intrinsic interest to homotopy theorists, its study is motivated by topological aspects of the motion planning problem in robotics. Define the topological complexity of a space X, denoted $\mathsf{TC}(X)$, to be the sectional category of the free path fibration $\pi_X\colon X^I \to X \times X$, which sends a path γ in X to its pair $(\gamma(0), \gamma(1))$ of initial and final points. The number $\mathsf{TC}(X)$ gives a quantitative measure of the 'navigational complexity' of X, when viewed as the configuration space of a mechanical system. Topological complexity is

☆ This work was partially supported by grants from the Simons Foundation: (#209575 to Gregory Lupton) and (#244393 to John Oprea).
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http://dx.doi.org/10.1016/j.topol.2015.04.005
0166-8641/

a close relative of the Lusternik–Schnirelmann category $\mathsf{cat}(X)$, although the two are independent. Further details and full definitions will be given in Section 2.

We remark once and for all that in this paper we adopt the convention of normalizing all category-type invariants to be one less than the number of open sets in the cover. So for instance, $\mathsf{TC}(X) = \mathsf{cat}(X) = 0$ when X is contractible.

Recall that a path-connected space X is *aspherical* if $\pi_i(X) = 0$ for $i \geq 2$. The homotopy type of an aspherical space is determined by the isomorphism class of its fundamental group. Furthermore, for any discrete group G one may construct, in a functorial way, a based aspherical complex $K(G, 1)$ having G as its fundamental group. Through this construction, any new homotopy invariant of spaces leads to a new and potentially interesting algebraic invariant of groups. In this paper we address the following problem, posed by Farber in [15]: Can one express $\mathsf{TC}(G) := \mathsf{TC}\big(K(G, 1)\big)$ in terms of more familiar invariants of the group G? This is an interesting open problem, about which relatively little is known beyond some particular cases (see below). In contrast, the corresponding problem for Lusternik–Schnirelmann category was solved in the late 1950s and early 1960s, with work of Eilenberg–Ganea [12], Stallings [26] and Swan [28]. Their combined work showed that $\mathsf{cat}(G) := \mathsf{cat}(K(G, 1)) = \mathrm{cd}(G)$, where cd denotes the *cohomological dimension*, a familiar algebraic invariant of discrete groups.

Groups G for which the precise value of $\mathsf{TC}(G)$ is known include: orientable surface groups [13]; pure braid groups $\mathcal{P}_n$ [18] and certain of their subgroups $\mathcal{P}_{n,m} = \ker(\mathcal{P}_n \to \mathcal{P}_m)$ which are kernels of homomorphisms obtained by forgetting strands [16]; right-angled Artin groups [6]; basis-conjugating automorphism groups of free groups [7]; and almost-direct products of free groups [5]. In all of these calculations, sharp lower bounds are given by cohomology with untwisted coefficients. If k is a field, let $\cup : H^*(G; k) \otimes H^*(G; k) \to H^*(G; k)$ denote multiplication in the cohomology k-algebra of the group G. The ideal $\ker(\cup) \subseteq H^*(G; k) \otimes H^*(G; k)$ is called the *ideal of zero-divisors*. One then has that $\mathsf{TC}(G) \geq \mathrm{nil}\ker(\cup)$, where nil denotes the nilpotency of an ideal. This is often referred to as the *zero-divisors cup-length* lower bound.

On the other hand, it is known that zero-divisors in cohomology with untwisted coefficients are not always sufficient to determine topological complexity. In [20] the topological complexity of the link complement of the Borromean rings was studied, and sectional category weight and Massey products were applied to obtain lower bounds. This is, to the best of our knowledge, the only previously known example of an aspherical space X for which $\mathsf{TC}(X)$ is greater than the zero-divisors cup-length for any field of coefficients.

In this paper we give new lower bounds for $\mathsf{TC}(G)$ which are described in terms of the subgroup structure of G. These lower bounds do not, therefore, require knowledge of the cohomology algebra of G or its cohomology operations.

Theorem 1.1. *Let G be a discrete group, and let A and B be subgroups of G. Suppose that $gAg^{-1} \cap B = \{1\}$ for every $g \in G$. Then $\mathsf{TC}(G) \geq \mathrm{cd}(A \times B)$.*

Thus $\mathsf{TC}(G)$ is bounded below by the cohomological dimension of the direct product $A \times B$ if no non-trivial element of A is conjugate in G to an element of B. Note that $A \times B$ is not a subgroup of G in general, and so it may well happen that $\mathrm{cd}(A \times B) > \mathrm{cd}(G)$.

From another viewpoint, Theorem 1.1 implies that upper bounds for the topological complexity of G force certain pairs of subgroups of G to contain non-trivial conjugate elements.

We next note some general settings in which the assumptions of Theorem 1.1 are satisfied. Recall that subgroups A and B of G are *complementary* if $A \cap B = \{1\}$ and $G = AB$. Then for every $g \in G$ we can write $g^{-1} = \alpha\beta$ for some $\alpha \in A$ and $\beta \in B$, and the condition $gAg^{-1} \cap B = \{1\}$ follows easily from $A \cap B = \{1\}$. In the special case when either A or B is normal in G, then G is a semi-direct product.