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Rigidity of topological duals of spaces of formal series with respect to product topologies

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To Daniel Barsky, on the occasion of his retirement

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ABSTRACT

Even in spaces of formal power series is required a topology in order to legitimate some operations, in particular to compute infinite summations. In general the topologies considered are just a product of the topology of the base field, an inverse limit topology or a topology induced by a pseudo-valuation. As our main result we prove the following phenomenon: the (left and right) topological dual spaces of formal power series equipped with the product topology with respect to any Hausdorff division ring topology on the base division ring, are all the same, namely just the space of polynomials. As a consequence, this kind of rigidity forces linear maps, continuous with respect to any (and then to all) those topologies, to be defined by very particular infinite matrices similar to row-finite matrices.

1. Introduction

Some manipulations of formal power series require some topological properties in order to be legitimate. For instance, the usual substitution of a power series in one variable without constant term into another, or the existence of the star operation, related to the Möbius inversion formula, are usually treated using either an order function (a pseudo-valuation) or, equivalently (while more imprecise), arguing that only finitely many terms contribute to the calculation in each degree (this is the usual sort of arguments used

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in combinatorics[2]). In both cases is used, explicitly or not, a topology induced by a filtration: the "order" of some partial sums must increase indefinitely for the sum to be defined (and the operation to be legal). Quite naturally other topologies may be used: for instance if X is an infinite set, then the completion of the algebra $R\langle X\rangle$ of polynomials in non-commutative variables (where R is a commutative ring with a unit) with respect to the usual filtration (induced by the length of a word in the free monoid X^*) is the set of all series with only finitely many non-zero terms for each given length. The sum $\sum_{x\in X} x$ of the alphabet (the "zêta function" of X) does not even exist in this completion. In order to take such series into account – which could be fundamental for instance in the case of the Möbius inversion formula – we must use the product topology (with a discrete R) or consider a graduation of X so that there are only finitely many members of X in each degree. If some analytical investigations must be performed (such as convergence ray, resolution of differential equations, or some functional analysis), the discrete topology of $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is not sufficient anymore; the absolute value of $\mathbb{K}$ turns to be unavoidable. Other topologies may be used for particular needs.

Given a topology, compatible in a natural way with algebraic operations, on a space of formal power series with coefficients in a topological field (or division ring), it can be useful to consider continuous linear endomorphisms because they commute to infinite sums. Quite amazingly for a very large class of admissible topologies (namely product topologies with respect to Hausdorff topologies on the base field or division ring) it appears that these continuous linear maps may be seen as infinite matrices of a particular kind (each "row" is finitely supported) and that, independently of the topology chosen for the base field. In other terms, a linear map can be represented as some "row-finite" matrix if it is continuous for one and thus for all these topologies (see Section 7). Hence, in order to prove that an endomorphism is continuous with respect to some topology, it suffices to prove this property for the more convenient topology in the class. Note however that the representability of an endomorphism by a row-finite matrix is not sufficient to guarantee that it is continuous, because the representation is not faithful. Nevertheless each row-finite matrix represents a continuous linear endomorphism.

The explanation of this phenomenon relies on the following property of rigidity: the topological left and right duals of a given space of formal series, equipped with the product topology, with coefficients in some Hausdorff topological division ring, are forced to be the space of polynomials, independently of the topology on the base division ring, as soon as it is Hausdorff. This is the main result of the paper, presented in Section 3 (Theorem 5) and proved in Section 4. We also recast this result with a more category-theoretic flavor (in Section 6) in order to show that it provides a natural equivalence between two categories of vector spaces, extending some results of J. Dieudonné on linearly compact vector spaces [7]. Some direct consequences of this property of rigidity, in particular the representation by row-finite matrices, are presented in Section 7.

A formal power series is a set-theoretic map defined on a free monoid and with values in a ring, represented as an infinite sum. Hence, from a linear perspective (i.e., ignoring the multiplication of series, or in other words, replacing the free monoid by any set), and a bit radically, the theory of formal power series matches with that of spaces of ring-valued functions. Within this point of view, a polynomial turns to be a finitely-supported map. Furthermore, as soon as a topology is considered for both the base ring (for instance the discrete topology) and the function space (the product topology) these maps may also be represented as sums of summable families (see Remark 14 below), whence formal power series are recovered, and nothing is lost through such a general viewpoint. Therefore, if the multiplication of series is irrelevant for the desired

[2] One observes that the topological aspects of combinatorics are often neglected, and sometimes ignored even in the most famous textbooks. I cite Stanley [19, p. 196]: "Algebraically inclined readers can think of $\mathbb{K}\langle\langle X\rangle\rangle$ [Author's note: the space of formal power series in non-commutative variables in the set X over $\mathbb{K}$] as the completion of the monoid algebra of the free monoid X^* with respect to the ideal generated by X." but this is of course false as soon as X is infinite, because in this case the series $\sum_{x\in X} x$ belongs to $\mathbb{K}\langle\langle X\rangle\rangle$ while it does not belong to the completion of the monoidal algebra $\mathbb{K}[X^*]$ with respect to the $\mathfrak{M}$-adic topology induced by its augmentation ideal $\mathfrak{M} = X\mathbb{K}[X^*]$.