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# A collection of nonfilling multicurve complexes

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ABSTRACT

We introduce a new collection of simplicial complexes associated to a connected orientable compact surface $S$, called $k$-curve complexes, denoted by $k$-$\mathcal{C}(S)$, each containing vertices given by $(k-1)$-simplices of the original curve complex of $S$, $\mathcal{C}(S)$ and edges given by a restricted nonfillingness property between vertices. We prove that each complex of this collection of complexes is connected and we study their coarse geometry, in particular, we prove that the first complex, 1-$\mathcal{C}(S)$, is hyperbolic and distances in $k$-$\mathcal{C}(S)$ are comparable to the distances in the marking complex and therefore comparable to the length of word path in $\mathcal{MCG}(S)$. In addition, we relate the combinatorial information of 1-$\mathcal{C}(S)$ with the topology of 3-manifolds and we provide an improved bound for distance of Heegaard splittings by using the distance in the 1-curve complex, 1-$\mathcal{C}(S)$.



## 1. Introduction

In 1978 Harvey [8], studying the action of the modular group at infinity on the Teichmüller space of a surface introduced a simplicial complex, which was analogous to the Tits buildings associated to semisimple Lie groups. This complex was later known as the Curve Complex of a surface and it has been a useful tool in the study of the topology of 3-manifolds. The curve complex is defined as the complex whose $\ell$-simplices correspond to sets of $\ell+1$ disjoint isotopy classes of essential simple closed curves on a surface $S$. On the other hand, Hatcher, in [9], introduced the Pants Complex, a simplicial complex where vertices are given by pants decompositions and edges between vertices by two moves provided by minimal intersection between two curves contained in the corresponding vertex. We call this simplicial complex the $k$-Curve Complex of $S$ and denote it by $k$-$\mathcal{C}(S)$

In the current work we generalize the idea of disjointness in order to interpolate the 1-skeleton of the curve complex and the pants graph of a given surface. So, for a fixed connected, orientable compact surface $S$ satisfying that the number $\xi(S) = 3g-3+n \geq 2$ and for a fixed number $k$, $1 \leq k \leq \xi - 2$, we introduce a

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new family of complexes. Each complex having multicurves of cardinality equal to $k$ as vertices, while edges between two nonisotopic vertices are given when two multicurves do not fill the surface $S$.

First, we prove some results for the case $k = 1$. Connectedness of the complex is proved using the idea, due by Lickorish, of doing a pair of local moves about the intersection points between two "curves" in the case of the original curve complex. However, these two moves can be generalized for the case when edges are given by nonfillingness instead disjointness for the next case $k = 2$.

**Theorem.** *Let $S$ be a connected, orientable compact surface satisfying that $\xi(S) \geq 2$. Then $1$-$\mathcal{C}(S)$ is a simplicial complex which is connected, hyperbolic, infinite-dimensional and of infinite diameter.*

For $k = 2$ we show connectedness for a closed surface using a pair of "generalized moves". The key feature about the proof of this result is that we find a path between two vertices making evident that we can use an algorithm to construct a path between any pair of vertices (corresponding to two different multicurves).

**Theorem.** *Let $S$ be a connected, orientable closed surface satisfying that $\xi(S) \geq 2$. Then the simplicial complex $2$-$\mathcal{C}(S)$ is connected.*

For the general case we use the hierarchy machinery developed by Masur and Minsky in [17] to show connectedness. And this procedure also shows the relation between the distance in $k$-$\mathcal{C}(S)$ and the distance in the marking complex of $S$ and therefore with the word path in the mapping class group $\mathcal{MCG}(S)$.

**Theorem.** *Let $S$ be a connected, orientable compact surface satisfying that $\xi(S) \geq 2$. Then the simplicial complex $k$-$\mathcal{C}(S)$ is connected, infinite dimensional and of infinite diameter. Also, for every $\alpha$, $\beta$ vertices in $k$-$\mathcal{C}(S)$ there exist constants $C$ and $D$ such that*

$$d_k(\alpha, \beta) < |H| < Cd_{\mathcal{M}}(\mu, \mu') + D$$

*where $\mu, \mu'$ are complete clean markings such that $\alpha \subset base(\mu)$ and $\beta \subset base(\mu')$.*

## 2. Preliminaries

### 2.1. Background on surfaces, curves and multicurves

Throughout this work a *surface* $S = S_{g,n}$ will denote a 2-dimensional manifold which is orientable, compact and connected, where $g$ denotes the genus of the surface and $n$ denotes the number of boundary components. We usually say that the surface $S = S_{g,n}$ is a *surface of type* $(g, n)$. We define the complexity of $S$ as the integer given by $\xi(S) := 3g - 3 + n \geq 2$.

A surface $S$ is called a *pair of pants* if the surface is homeomorphic to a 3-punctured 2-sphere, i.e., $S = S_{0,3}$. It is a well-known fact that for every surface $S = S_{g,n}$ with $\xi(S) \geq 1$ there exists a collection of $3g - 3 + n$ curves such that they cut $S_{g,n}$ into a collection of pair of pants. A collection of curves having this property is called a *maximal collection of curves*.

**Definition 2.1.** A *(pair of) pants decomposition* $P$ of a surface $S$ is a collection of disjoint essential simple closed curves in $S$ such that $S - P =$ 'union of pair of pants'.

Equivalently, a pants decomposition is a maximal collection of essential simple closed curves such that no two curves in the collection are isotopic and none is $\partial$-parallel or bounds a disk (see Fig. 1).

Any surface $S = S_{g,n}$ with $\xi(S) \geq 2$ or $S = S_{0,4}$ have an infinite number of pants decompositions and as Hatcher and Thurston showed in [10] it is possible to define a simplicial complex out of this collection of