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Free products of groups with bi-invariant metrics

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ABSTRACT

For a family of groups G_λ which are equipped with bi-invariant metrics, we construct, in a very explicit manner, a bi-invariant metric on the free product of the G_λ. The universal properties of this construction are investigated and some open questions are formulated.

Given a family $(G_\lambda)_{\lambda\in\Lambda}$ of groups with bi-invariant metrics, the main purpose of the present note is to define a bi-invariant metric on the free product $\underset{\lambda\in\Lambda}{*} G_\lambda$ and to investigate its universal properties.

For a group G, a map $\| \,.\, \| : G \to \mathbb{R}_{\geq 0}$ is called a *norm* on G if it satisfies the following properties:

(1) $\| g \| = 0 \Leftrightarrow g = 1$.
(2) $\| g^{-1} \| = \| g \|$.
(3) $\| gh \| = \| hg \|$ or, equivalently, $\| ghg^{-1} \| = \| h \|$.
(4) $\| gh \| \leq \| g \| + \| h \|$.

It will be more convenient to consider norms than bi-invariant metrics. Given a norm, a bi-invariant metric is obtained by $d(g,h) = \| gh^{-1} \|$, and given a bi-invariant metric, a norm is obtained by $\| g \| = d(g,1)$.

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Starting from norms on the groups G_λ, we want to define a norm on $G := \mathop{*}\limits_{\lambda\in\Lambda} G_\lambda$. We assume that the sets $G_\lambda \setminus \{1\}$ are pairwise disjoint. Let us write

$$\overline{G} := \prod_{\lambda\in\Lambda} G_\lambda,$$
$$\overline{\overline{G}} := \overline{G} \times \overline{G} \times \overline{G} \times \ldots .$$

Elements of $\overline{\overline{G}}$ are written in the form

$$x = (x_{n,\lambda})_{n\in\mathbb{N},\lambda\in\Lambda} = (x_a)_{a\in\mathbb{N}\times\Lambda} \quad \text{with } x_{n,\lambda} \in G_\lambda;$$

the *support* of x is given by

$$\operatorname{supp} x := \{a \in \mathbb{N}\times\Lambda \mid x_a \neq 1\} .$$

We put

$$\tilde{G} := \{x \in \overline{\overline{G}} \mid \operatorname{supp} x \text{ is finite}\} .$$

For the moment, we choose a total ordering on Λ and order the set $\mathbb{N}\times\Lambda$ lexicographically, i.e.

$$(n,\lambda) < (m,\mu) \Leftrightarrow n < m \text{ or } (n = m \text{ and } \lambda < \mu) .$$

Definition 1. An element $x \in \tilde{G}$ is called *reducible* if there is $(n,\lambda) \in \mathbb{N}\times\Lambda$ such that:

- $x_{m,\mu} = 1$ for $(n,\lambda) < (m,\mu) < (n+1,\lambda)$.
- There exists a pair $(m,\mu) \geq (n+1,\lambda)$ with $x_{m,\mu} \neq 1$.

We then define $y \in \tilde{G}$ by

$$y_{m,\mu} := \begin{cases} x_{m,\mu} & \text{for} \quad (m,\mu) < (n,\lambda), \\ x_{n,\lambda}\cdot x_{n+1,\lambda} & \text{for} \quad (m,\mu) = (n,\lambda), \\ x_{m+1,\mu} & \text{for} \quad (m,\mu) > (n,\lambda), \end{cases}$$

and y is said to be obtained from x *by cancellation.*

Remark 1. Given $x \in \tilde{G}$, there is a finite sequence of cancellations such that the resulting element $\bar{x}$ is no longer reducible. The element $\bar{x}$ is determined by x; it doesn't depend on the sequence of cancellations.

There is an obvious surjection

$$\pi : \tilde{G} \to G .$$

Its restriction to the set of irreducible elements is a bijection. In order to define a norm on G, we shall construct a map

$$\| \,.\, \| : \tilde{G} \to \mathbb{R}$$

which is compatible with cancellation. This map will be defined in terms of decompositions of $\mathbb{N}\times\Lambda$.