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Catalan states of lattice crossing

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For a lattice crossing $L(m,n)$ we show which Catalan connection between $2(m+n)$ points on the boundary of $m \times n$ rectangle P can be realized as a Kauffman state and we give an explicit formula for the number of such Catalan connections. For the case of a Catalan connection with no arc starting and ending on the same side of the tangle, we find a closed formula for its coefficient in the Relative Kauffman Bracket Skein Module of $P \times I$.

1. Introduction

For an oriented 3-manifold M^3, *Kauffman Bracket Skein Module* (*KBSM*) was defined by J.H. Przytycki in [8]. If $F_{g,n}$ denotes an oriented surface of genus g with n boundary components and $I = [0,1]$, *KBSM* of $M^3 = F_{g,n} \times I$ has a natural structure of an algebra $S_{2,\infty}(M^3)$ (called the *Skein Algebra* of M^3), where the multiplication is defined by placing link L_1 above link L_2. Since results concerning the multiplicative structure of $S_{2,\infty}(M^3)$ play an important role in the study of quantizations of $\mathrm{SL}(2,\mathbb{C})$-character varieties of fundamental groups of surfaces, in several important cases $(g,n) \in \{(1,0),(1,1),(1,2)\}$ the multiplicative structure on $S_{2,\infty}(F_{g,n} \times I)$ has been very well studied and understood (see for instance [1], Theorem 2.1, Corollary 2.2, and Theorem 2.2). Moreover, for $M^3 = F_{0,4} \times I$, the presentation for $S_{2,\infty}(M^3)$ was also obtained in [1] (see Theorem 3.1) and a very elegant formula for the product in $S_{2,\infty}(F_{1,0} \times I)$ was found by

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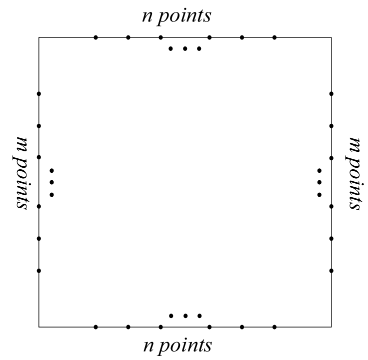

Fig. 1.1. Parallelogram P.

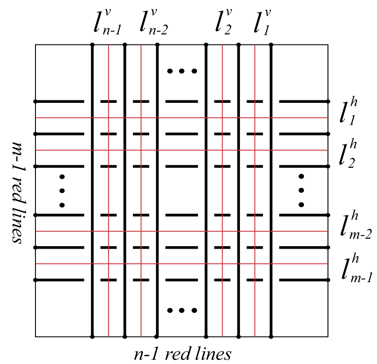

Fig. 1.2. $(m+n)$-tangle $L(m,n)$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

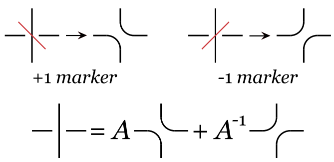

Fig. 1.3. $+1$ or -1 markers.

R. Gelca and C. Frohman in [3]. Being encouraged by results of [3], we started our quest to find a formula of the similar-type for the product in $S_{2,\infty}(F_{0,4} \times I)$. Therefore, this work might be viewed as a first step in this direction that concentrates on the local analysis of the problem.

We consider the Relative Kauffman Bracket Skein Module[1] (*RKBSM*) of $P \times I$, where P is an $m \times n$ parallelogram with $2(m+n)$ points on the boundary arranged as shown in Fig. 1.1, and $(m+n)$-tangle $L(m,n)$ shown in Fig. 1.2 that we will refer to as an $m \times n$*-lattice crossing*. If $\mathfrak{Cat}_{m,n}$ denotes the set of all Catalan states for P (crossingless connections between boundary points) then $L(m,n)$ in *RKBSM* of $P \times I$ can be uniquely written in the form $L(m,n) = \sum_{C \in \mathfrak{Cat}_{m,n}} r(C)C$, where $r(C) \in \mathbb{Z}[A^{\pm 1}]$.

Let $\mathcal{S}_{m,n}$ be the set of all Kauffman states (choices of positive or negative markers for all crossings as shown in Fig. 1.3), after applying skein relations we have in the *RKBSM* of $P \times I$:

$$L(m,n) = \sum_{s \in \mathcal{S}_{m.n}} A^{p(s)-n(s)} \left(-A^2 - A^{-2}\right)^{|s|} K(s)$$

[1] *RKBSM* was defined in [8] and it was noted there that the *RKBSM* for $D^2 \times I$ with $2k$ points on the boundary is a free module with the basis consisting of Catalan connections. For an extensive discussions of theory of Kauffman bracket skein modules see [9].